Assignment #21

Due on Friday, November 21, 2008

Read on *The Fundamental Theorem of Calculus,* pp. 279–320 in Chapter 10 of Bressoud's book.

Background and Definitions

Green's Theorem. The Fundamental Theorem of Calculus,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega,$$

takes the following form in two–dimensional Euclidean space:

Let R denote a region in $\mathbb{R}^2$ bounded by a simple closed curve, $\partial\mathbb{R}$, made up of a finite number of C^1 paths traversed in the counterclockwise sense. Let P and Q denote two C^1 scalar fields defined on some open set containing R and its boundary, ∂R. Then,

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \int_{\partial R} P\mathrm{d}x + Q\mathrm{d}y. \tag{1}$$

Do the following problems

1. Apply Green's Theorem, as expressed in the formula (1), to the functions $P(x,y) = -y$ and $Q(x,y) = x$ to derive the formula

$$\text{area}(R) = \frac{1}{2}\int_{\partial R} -y\mathrm{d}x + x\mathrm{d}y. \tag{2}$$

to compute the area of the region R.

2. Use the formula (2) derived in the previous theorem to compute the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive real numbers.

3. In Problem 1(b) of Assignment #20, you showed that another form of Fundamental Theorem of Calculus in two dimensions is

$$\int_R \operatorname{div}(F)\ \mathrm{d}x\mathrm{d}y = \text{Flux of } F \text{ across } \partial R,$$

where $\operatorname{div}(F) = \dfrac{\partial P}{\partial x} + \dfrac{\partial P}{\partial y}$ is the divergence of the vector field $F = P\widehat{i} + Q\widehat{j}$; that is, the flux of F across the boundary of R is the double integral of the divergence of F over the region R. Thus, the Fundamental Theorem of Calculus in $\mathbb{R}^2$ takes the form

$$\int_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \int_{\partial R} F \cdot \widehat{n}\ \mathrm{d}s, \tag{3}$$

where $\widehat{n}$ is a unit vector perpendicular to ∂R and pointing to the outside of ∂R.

Use formula (3) to compute the flux of the field $F = x\,\widehat{i} + y\,\widehat{j}$ across the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

4. Let f and g be two scalar fields defined on some open subset of $\mathbb{R}^2$. Suppose that f and g are C^1 and that ∇g is a C^1 vector field. Show that

$$\operatorname{div}(f\nabla g) = \nabla f \cdot \nabla g + f\operatorname{div}(\nabla g).$$

$\operatorname{div}(\nabla g)$ is called the *Laplacian* of g and is usually denoted by Δg; thus,

$$\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}.$$

5. Let f and g be as in the previous problem. Use formula (3) and the result from the previous problem to show that

$$\int_R f\Delta g\ \mathrm{d}x\mathrm{d}y = \int_{\partial R} f\frac{\partial g}{\partial n}\ \mathrm{d}s - \int_R \nabla f \cdot \nabla g\ \mathrm{d}x\mathrm{d}y,$$

where $\dfrac{\partial g}{\partial n}$ denotes the derivative of g in the direction of $\widehat{n}$, or $D_{\widehat{n}}g$.