Assignment #22

Due on Monday, November 24, 2008

Read on *The Fundamental Theorem of Calculus,* pp. 279–320 in Chapter 10 of Bressoud's book.

Background and Definitions
Recall that the Fundamental Theorem of Calculus,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega,$$

takes the following two forms in two–dimensional Euclidean space:

Let R denote a region in $\mathbb{R}^2$ bounded by a simple closed curve, $\partial\mathbb{R}$, made up of a finite number of C^1 paths traversed in the counterclockwise sense. Let P and Q denote two C^1 scalar fields defined on some open set containing R and its boundary, ∂R.

Theorem 0.1 (Green's Theorem)**.** *Then,*

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial R} P\,dx + Q\,dy. \tag{1}$$

and, if $F = P\widehat{i} + Q\widehat{j}$,

Theorem 0.2 (The Divergence Theorem)**.** *Then,*

$$\int_R \mathrm{div} F \;\, dx\,dy = \oint_{\partial R} F \cdot \widehat{n} \;\, ds, \tag{2}$$

where $\widehat{n}$ is the outward unit normal to $\partial\mathbb{R}$.

Do the following problems

1. Use the Fundamental Theorem of Calculus in two dimensions to evaluate the line integral

$$\oint_C (y^2 + x^3)\mathrm{d}x + x^4\mathrm{d}y,$$

where C is the boundary of the unit square in $\mathbb{R}^2$,

$$[0,1] \times [0,1] = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$$

traversed in the counterclockwise direction.

2. Use the Fundamental Theorem of Calculus in two dimensions to evaluate the line integral $\oint_C (x^2 + x^3)\mathrm{d}x + y^4\mathrm{d}y$, where C is any simple, closed, C^1 curve.

3. Let $P(x, y)$ and $Q(x, y)$ denote C^1 functions defined on an open subset D of $\mathbb{R}^2$. Show that the divergence, $\mathrm{div}F$, of the vector field

$$F(x, y) = P(x, y)\widehat{i} + Q(x, y)\widehat{j}, \quad \text{for all } (x, y) \in D,$$

is continuous on D. In particular, deduce that given (x_o, y_o), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_o| < \delta \ \text{ and } \ |y - y_o| < \delta \Rightarrow |\mathrm{div}F(x, y) - \mathrm{div}F(x_o, y_o)| < \varepsilon.$$

4. Let P, Q and F be as in Problem 3.

Fix $(x_o, y_o) \in D$. Given $\delta > 0$, define the square region, R_δ, around (x_o, y_o) to be

$$R_\delta = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x_o - \frac{\delta}{2} \leqslant x \leqslant x_o \frac{\delta}{2}, y_o - \frac{\delta}{2} \leqslant y \leqslant y_o \frac{\delta}{2} \right\}.$$

Denote by ∂R_δ the boundary of the square R_δ traversed in the counterclockwise direction.

Use the Fundamental Theorem of Calculus to evaluate the flux of F across the boundary of the square R_δ; that is, evaluate

$$\oint_{\partial R_\delta} F \cdot \widehat{n} \, \mathrm{d}s,$$

where $\widehat{n}$ is the outward unit normal to ∂R_δ wherever it is defined.

5. Let P, Q and F be as in Problem 3, and $\mathbb{R}_\delta$ be as in Problem 4.

Show that

$$\lim_{\delta \to 0} \left(\frac{1}{\delta^2} \oint_{\partial R_\delta} F \cdot \widehat{n} \, \mathrm{d}s \right) = \mathrm{div}F(x_o, y_o).$$

Give and interpretation of this result.

Suggestion: Consider

$$\left| \frac{1}{\delta^2} \int_{R_\delta} \mathrm{div}F(x, y) \, \mathrm{d}x\mathrm{d}y - \mathrm{div}F(x_o, y_o) \right|$$

and note that

$$\mathrm{div}F(x_o, y_o) = \frac{1}{\delta^2} \int_{R_\delta} \mathrm{div}F(x_o, y_o) \, \mathrm{d}x\mathrm{d}y.$$

Use the result of Problem 3.