Review Problems for Exam 2

1. Consider a wheel of radius a which is rolling on the x–axis in the xy–plane. Suppose that the center of the wheel moves in the positive x–direction and a constant speed v_o. Let P denote a fixed point on the rim of the wheel.

   (a) Give a path $\sigma(t) = (x(t), y(t))$ giving the position of the P at any time t, if P is initially at the point $(0, 2a)$.

   (b) Compute the velocity of P at any time t. When is the velocity of P horizontal? What is the speed of P at those times?

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,t) = f(x - ct) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2,$$

   where c is a real constant.

   Show that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,y) = f(r) \quad \text{where} \quad r = \sqrt{x^2 + y^2} \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

   Express the Laplacian of u, Δu, i.e., the divergence of the gradient of u, in terms of f', f'' and r.

4. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.

   (a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.

   (b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.

   (c) Find largest and the smallest value of f on C.

5. Let $C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$; i.e., C is the upper unit semi–circle. C can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1 - \tau^2}) \quad \text{for} \quad -1 \leqslant \tau \leqslant 1.$$

   (a) Compute $s(t)$, the arclength along C from $(-1, 0)$ to the point $\sigma(t)$, for $0 \leqslant t \leqslant 1$.

(b) Compute $s'(t)$ for $-1 < t < t$ and sketch the graph of s as function of t.

(c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leqslant t \leqslant 1$, and deduce that

$$\sin(s(t)) = \sqrt{1-t^2} \quad \text{for all} \quad -1 \leqslant t \leqslant 1.$$

6. Let R denote the open unit disc in $\mathbb{R}^2$; that is, $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Evaluate the integral
$$\int_R \ln(x^2+y^2) \ \ \mathrm{d}x \ \mathrm{d}y$$
by first evaluating the integral
$$\int_{A_\varepsilon} \ln(x^2+y^2) \ \ \mathrm{d}x \ \mathrm{d}y,$$
where A_ε is the annulus $\{(x,y) \in \mathbb{R}^2 \mid \varepsilon^2 < x^2 + y^2 < 1\}$, for $0 < \varepsilon < 1$, and then computing the limit at ε goes to 0.

7. Let A denote the annulus $\{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$, and evaluate $\displaystyle\int_A \frac{1}{x^2+y^2} \ \ \mathrm{d}x \ \mathrm{d}y$.

8. Let $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant y, x^2 + y^2 \leqslant 1\}$, and evaluate $\displaystyle\int_R x^2 \ \ \mathrm{d}x \ \mathrm{d}y$.

9. Let R denote the region in the xy–plane bounded by the lines $x + y = 1$, $x + y = 4$, $x - y = -1$ and $x - y = 1$. Evaluate $\displaystyle\int_R (x+y)e^{x-y} \ \ \mathrm{d}x \ \mathrm{d}y$.

10. Evaluate $\displaystyle\int_R (x+y) \ \ \mathrm{d}x \ \mathrm{d}y$ where R is the rectangle in the xy–plane with vertices $(1,0)$, $(4,3)$, $(3,4)$ and $(0,1)$.

11. Evaluate $\displaystyle\int_R (x-y) \ \ \mathrm{d}x \ \mathrm{d}y$ where R is the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$.

12. Let $\omega = 2x \ \ \mathrm{d}x + y \ \ \mathrm{d}y$ and $\eta = y \ dx - x \ \ \mathrm{d}y$ denote differential 1–forms. Compute each of the following $\omega \ \ \mathrm{d}\eta$, $\eta \ \ \mathrm{d}\omega$ and $\mathrm{d}(\omega\eta)$.

13. Let C denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral $\displaystyle\int_C x^3 \ \ \mathrm{d}y - y^3 \ \mathrm{d}x$.

14. Let $F(x,y) = y\,\widehat{\imath} - x\,\widehat{\jmath}$ and R be the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\displaystyle\int_{\partial R} F \cdot n \ \ \mathrm{d}s$.