Review Problems for Exam 2

1. Consider a wheel of radius a which is rolling on the x–axis in the xy–plane. Suppose that the center of the wheel moves in the positive x–direction and a constant speed v_o. Let P denote a fixed point on the rim of the wheel.

   (a) Give a path $\sigma(t) = (x(t), y(t))$ giving the position of the P at any time t, if P is initially at the point $(0, 2a)$.

   ***Solution**:* Let $\theta(t)$ denote the angle that the ray from the center

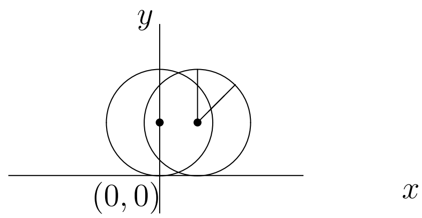

Figure 1: Circle

   of the circle to the point $(x(t), y(t))$ makes with a vertical line through the center. Then, $v_o t = a\theta(t)$; so that $\theta(t) = \dfrac{v_o}{a}t$ and

$$x(t) = v_o t + a\sin(\theta(t))$$

   and

$$y(t) = a + a\cos(\theta(t))$$

$\square$

   (b) Compute the velocity of P at any time t. When is the velocity of P horizontal? What is the speed of P at those times?

   ***Solution**:* The velocity vector is

$$\sigma'(t) = (x'(t), y'(t)) = (v_o + a\theta'(t)\cos(\theta(t)), -a\theta'(t)\sin(\theta(t)))$$

   where

$$\theta'(t) = \frac{v_o}{a}.$$

   We then have that

$$\sigma'(t) = (v_o + v_o\cos(\theta(t)), -v_o\sin(\theta(t)))$$

The velocity of P is horizontal when

$$\sin(\theta(t)) = 0,$$

or

$$\theta(t) = n\pi,$$

where n is an integer; and when

$$\cos(\theta(t)) \neq -1.$$

We then get that the velocity of P is horizontal when

$$\theta(t) = 2k\pi$$

where k is an integer.
The speed at the points where the velocity if horizontal is then equal to $2v_o$. □

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,t) = f(x-ct) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2,$$

where c is a real constant.

Show that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

***Solution**:* Use the Chain Rule to compute

$$\frac{\partial u}{\partial t} = f'(x-ct)\cdot\frac{\partial}{\partial t}(x-ct) = -c\ f'(x-ct),$$

and

$$\frac{\partial^2 u}{\partial t^2} = c\ f''(x-ct)\cdot\frac{\partial}{\partial t}(x-ct) = c^2\ f''(x-ct).$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ct)$$

since $\dfrac{\partial}{\partial x}(x-ct) = 1$. Hence,

$$\frac{\partial^2 u}{\partial t^2} = c^2\ f''(x-ct) = c^2 \frac{\partial^2 u}{\partial x^2}.$$

□

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,y) = f(r) \quad \text{where} \quad r = \sqrt{x^2+y^2} \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

Express the Laplacian of u, Δu, i.e., the divergence of the gradient of u, in terms of f', f'' and r.

***Solution**:* First note that $r^2 = x^2 + y^2$, from which we get that

$$2r\frac{\partial r}{\partial x} = 2x,$$

or

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

Next, use the Chain Rule to compute

$$\frac{\partial u}{\partial x} = f'(r)\cdot\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}.$$

Differentiating with respect to x again, using the Chain, Product and Quotient rules,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(x\frac{f'(r)}{r}\right) \\
&= \frac{f'(r)}{r} + x\frac{\partial}{\partial x}\left(\frac{f'(r)}{r}\right) \\
&= \frac{f'(r)}{r} + x\frac{rf''(r)\frac{x}{r} - f'(r)\frac{x}{r}}{r^2} \\
&= \frac{f'(r)}{r} + \frac{x^2}{r^2}f''(r) - \frac{x^2}{r^3}f'(r)
\end{aligned}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + \frac{y^2}{r^2}f''(r) - \frac{y^2}{r^3}f'(r).$$

Hence

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= 2\frac{f'(r)}{r} + \frac{x^2+y^2}{r^2} f''(r) - \frac{x^2+y^2}{r^3} f'(r) \\ &= 2\frac{f'(r)}{r} + \frac{r^2}{r^2} f''(r) - \frac{r^2}{r^3} f'(r) \\ &= 2\frac{f'(r)}{r} + f''(r) - \frac{1}{r} f'(r) \\ &= f''(r) + \frac{1}{r} f'(r).\end{aligned}$$

□

4. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.

   (a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.

   (b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.

   ***Solution**:* The curve C is given by the set of points (x,y) in $\mathbb{R}^2$ such that

   $$2x^2 + y^2 = 1,$$

   or

   $$\frac{x^2}{1/2} + y^2 = 1.$$

   That is, C is an ellipse with minor vertices $\pm 1/\sqrt{2}$ and major vertices ± 1. □

   The sketch is shown in Figure 2.

   (c) Find largest and the smallest value of f on C.

   ***Solution**:* Let $\sigma(t)$ be a parametrization of the ellipse. We want to find a value of t for which the function $f(\sigma(t))$ is the largest. Thus, we first look for critical points of this function. By the Chain Rule,

   $$\frac{d}{dt}\left(f(\sigma(t))\right) = \nabla f(\sigma(t)) \cdot \sigma'(t).$$

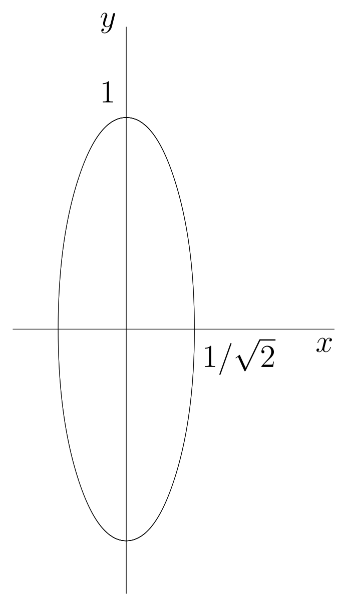

Figure 2: Sketch of ellipse

Thus, t is a critical point if the tangent vector $\sigma'(t)$ is perpendicular to $\nabla f(x, y) = 4\widehat{i} - 7\widehat{j}$.
On the other hand, from

$$g(\sigma(t)) = 1 \quad \text{for all t}$$

we get that

$$\nabla g(\sigma(t)) \cdot \sigma'(t) = 0$$

so that $\sigma'(t)$ is also perpendicular to $\nabla g(x, y) = 4x\widehat{i} + 2y\widehat{j}$. Hence, ∇f and ∇g must be parallel at a critical points; that is, there must be a constant $\lambda \neq 0$ such that

$$\nabla g(x, y) = \lambda \nabla f(x, y)$$

or

$$4x\widehat{i} + 2y\widehat{j} = 4\lambda\widehat{i} - 7\lambda\widehat{j}.$$

We then get that

$$4x = 4\lambda$$

and

$$2y = -7\lambda.$$

In other words, a critical point (x, y) must lie in the line

$$2y = -7x.$$

Next, we find the intersection of this line with the ellipse. Solving for y and substituting into the equation of the ellipse we get that

$$2x^2 + \left(\frac{-7x}{2}\right)^2 = 1$$

or

$$2x^2 + \frac{49}{4}x^2 = 1$$

or

$$\frac{57}{4}x^2 = 1$$

or

$$x^2 = \frac{4}{57}$$

from which we get that

$$x = \pm\frac{2}{\sqrt{57}}.$$

We therefore get the critical points

$$\left(\frac{2}{\sqrt{57}}, -\frac{7}{\sqrt{57}}\right) \quad \text{and} \quad \left(-\frac{2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right).$$

Evaluating f at each of these points we find that

$$f\left(\frac{2}{\sqrt{57}}, -\frac{7}{\sqrt{57}}\right) = \frac{8}{\sqrt{57}} + \frac{49}{\sqrt{57}} = \sqrt{57}$$

and

$$f\left(-\frac{2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right) = -\frac{8}{\sqrt{57}} - \frac{49}{\sqrt{57}} = -\sqrt{57}.$$

Thus, f is the largest at $\left(\frac{2}{\sqrt{57}}, -\frac{7}{\sqrt{57}}\right)$ and the smallest at $\left(-\frac{2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right)$. The largest value of f on C is then $\sqrt{57}$, and its smallest value on C is $-\sqrt{57}$. □

5. Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$; i.e., C is the upper unit semi–circle. C can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1-\tau^2}) \quad \text{for} \quad -1 \leqslant \tau \leqslant 1.$$

(a) Compute $s(t)$, the arclength along C from $(-1,0)$ to the point $\sigma(t)$, for $-1 \leqslant t \leqslant 1$.

Solution: Compute $\sigma'(\tau) = \left(1, -\dfrac{\tau}{\sqrt{1-\tau^2}}\right)$. for all $\tau \in (-1,1)$. Then,

$$\|\sigma'(\tau)\| = \sqrt{1 + \frac{\tau^2}{1-\tau^2}} = \frac{1}{\sqrt{1-\tau^2}}.$$

It then follows that

$$s(t) = \int_{-1}^{t} \frac{1}{\sqrt{1-\tau^2}}\, \mathrm{d}\tau \quad \text{for } -1 \leqslant t \leqslant 1.$$

□

(b) Compute $s'(t)$ for $-1 < t < t$ and sketch the graph of s as function of t.

Solution: By the Fundamental Theorem of Calculus,

$$s'(t) = \frac{1}{\sqrt{1-t^2}} \quad \text{for } -1 < t < 1.$$

Note then that $s'(t) > 0$ for all $t \in (-1,1)$ and therefore s is strictly increasing on $(-1,1)$.
Next, compute the derivative of $s'(t)$ to get the second derivative of $s(t)$:

$$s''(t) = \frac{t}{(1-t^2)^{3/2}} \quad \text{for } -1 < t < 1.$$

It then follows that $s''(t) < 0$ for $-1 < t < 0$ and $s''(t) > 0$ for $0 < t < 1$. Thus, the graph of $s = s(t)$ is concave down on $(-1,0)$ and concave up on $(0,1)$.
Finally, observe that $s(-1) = 0$, $s(0) = \pi/2$ (the arc–length along a quarter of the unit circle), and $s(1) = \pi$ (the arc–length along a semi–circle of unit radius). We can then sketch the graph of $s = s(t)$ as shown in Figure 3. □

(c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leqslant t \leqslant 1$, and deduce that

$$\sin(s(t)) = \sqrt{1-t^2} \quad \text{for all} \quad -1 \leqslant t \leqslant 1.$$

Solution: Figure 4 shows the upper unit semicircle and a point $\sigma(t)$ on it. Putting $\theta(t) = \pi - s(t)$, then $\theta(t)$ is the angle, in radians, that the ray from the origin to $\sigma(t)$ makes with the positive x–axis. It then follows that

$$\cos(\theta(t)) = t$$

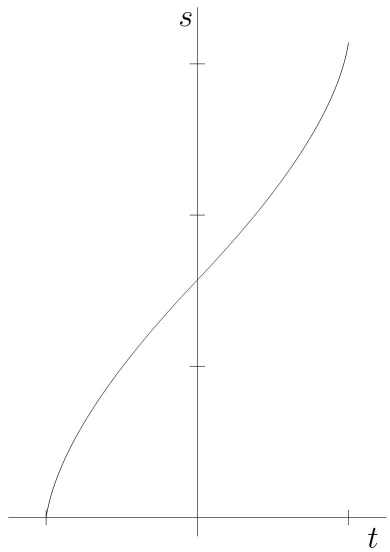

Figure 3: Sketch of $s = s(t)$

and

$$\sin(\theta(t)) = \sqrt{1 - t^2}.$$

Since

$$\sin(\theta(t)) = \sin(\pi - s(t)) = \sin(s(t),$$

the result follows.

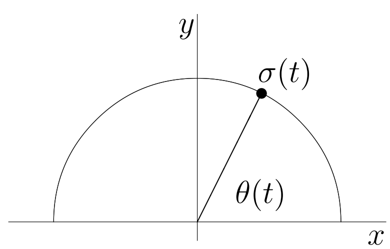

Figure 4: Sketch of Semi–circle

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6. Let R denote the open unit disc in $\mathbb{R}^2$; that is, $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Evaluate the integral

$$\int_R \ln(x^2 + y^2) \, \mathrm{d}x\mathrm{d}y$$

by first evaluating the integral

$$\int_{A_\varepsilon} \ln(x^2+y^2)\ \mathrm{d}x\mathrm{d}y,$$

where A_ε is the annulus $\{(x,y)\in\mathbb{R}^2 \mid \varepsilon^2 < x^2+y^2 < 1\}$, for $0<\varepsilon<1$, and then computing the limit at ε goes to 0.

Solution: First, evaluate the integral on the annulus pictured in Figure 5.

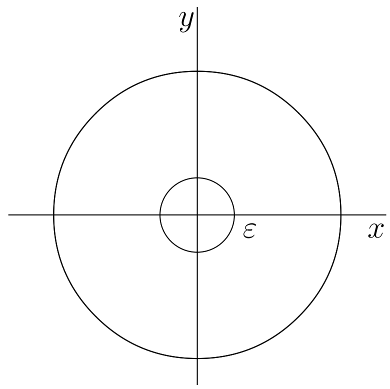

Figure 5: Sketch of A_ε

Using polar coordinates we obtain

$$\begin{aligned}\int_{A_\varepsilon} \ln(x^2+y^2)\ \mathrm{d}x\mathrm{d}y &= \int_0^{2\pi}\int_\varepsilon^1 \ln(r^2)\ r\ \mathrm{d}r\mathrm{d}\theta \\ &= 4\pi\int_\varepsilon^1 \ln(r)\ r\ \mathrm{d}r\end{aligned}$$

Integrating by parts, with $u=\ln(r)$ and $\mathrm{d}v = r\ \mathrm{d}r$, we then get that

$$\begin{aligned}\int_{A_\varepsilon} \ln(x^2+y^2)\ \mathrm{d}x\mathrm{d}y &= 4\pi\left[-\frac{\varepsilon^2}{2}\ln(\varepsilon) - \int_\varepsilon^1 \frac{r}{2}\ \mathrm{d}r\right] \\ &= 4\pi\left[\frac{\varepsilon^2}{4} - \frac{\varepsilon^2}{2}\ln(\varepsilon) - \frac{1}{4}\right].\end{aligned}$$

As $\varepsilon\to 0$, we can show, using L'Hospital's rule, that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{2}\ln(\varepsilon) = 0.$$

It then follows that

$$\int_R \ln(x^2+y^2)\, \mathrm{d}x\mathrm{d}y \;=\; \lim_{\varepsilon\to 0}\int_{A_\varepsilon} \ln(x^2+y^2)\, \mathrm{d}x\mathrm{d}y = -\pi.$$

□

7. Let A denote the annulus $\{(x,y)\in\mathbb{R}^2 \mid 1 < x^2+y^2 < 4\}$, and evaluate $\displaystyle\int_A \frac{1}{x^2+y^2}\, \mathrm{d}x\mathrm{d}y$.

***Solution**:* Proceeding as in the previous problem, we obtain that

$$\int_A \frac{1}{x^2+y^2}\, \mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\int_1^2 \frac{1}{r^2}\, r\, \mathrm{d}r\mathrm{d}\theta,$$

see Figure 6.

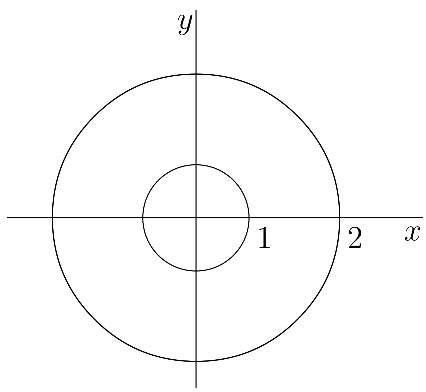

Figure 6: Sketch of A

Then,

$$\int_A \frac{1}{x^2+y^2}\, \mathrm{d}x\mathrm{d}y \;=\; 2\pi\int_1^2 \frac{1}{r}\, \mathrm{d}r = 2\pi\ln(2).$$

□

8. Let $R=\{(x,y)\in\mathbb{R}^2 \mid 0\leqslant x\leqslant y, x^2+y^2\leqslant 1\}$, and evaluate $\displaystyle\int_R x^2\, \mathrm{d}x\mathrm{d}y$.

***Solution**:* A sketch of the region R is shown in Figure

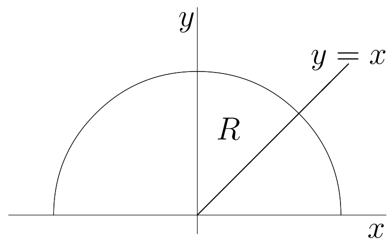

Figure 7: Sketch of region R in Problem 8

We may use polar coordinates to do this problem as follows

$$\begin{aligned}\int_R x^2 \, \mathrm{d}x\mathrm{d}y &= \int_{\pi/4}^{\pi/2}\int_0^1 r^2\cos^2(\theta)\; r\; \mathrm{d}r\mathrm{d}\theta \\ &= \int_{\pi/4}^{\pi/2}\cos^2(\theta)\int_0^1 r^3\;\mathrm{d}r\mathrm{d}\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2}\cos^2(\theta)\;\mathrm{d}\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2}\frac{1}{2}(1+\cos(2\theta))\;\mathrm{d}\theta,\end{aligned}$$

where we have used the double angle formula for $\cos^2\theta$. We then have that

$$\begin{aligned}\int_R x^2 \, \mathrm{d}x\mathrm{d}y &= \frac{1}{8}\int_{\pi/4}^{\pi/2}(1+\cos(2\theta))\;\mathrm{d}\theta \\ &= \frac{1}{8}\left[\theta+\frac{1}{2}\sin(2\theta)\right]_{\pi/4}^{\pi/2} \\ &= \frac{1}{8}\left(\frac{\pi}{4}-\frac{1}{2}\right) \\ &= \frac{1}{32}\left(\pi-2\right).\end{aligned}$$

□

9. Let R denote the region in the xy–plane bounded by the lines $x+y=1$, $x+y=4$, $x-y=-1$ and $x-y=1$. Evaluate $\displaystyle\int_R (x+y)e^{x-y}\,\mathrm{d}x\mathrm{d}y$.

***Solution**:* The region for this problem is sketched in Figure 8.

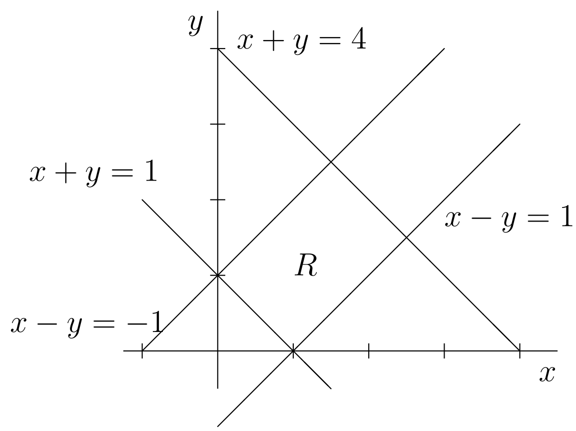

Figure 8: Sketch of Region R in Problem 9

Make the change of variables $x+y=u$ and $x-y=v$. We then obtain that

$$\begin{aligned} x &= \frac{1}{2}u + \frac{1}{2}v \\ y &= \frac{1}{2}u - \frac{1}{2}v \end{aligned}$$

We then obtain the change of coordinates map

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

which maps the rectangle $D=\{(u,v)\in\mathbb{R}^2 \mid 1\leqslant u\leqslant 4, -1\leqslant v\leqslant 1\}$ to the region R. The change of variables formula then yields

$$\int_R (x+y)e^{x-y}\,\mathrm{d}x\mathrm{d}y = \int_D ue^v\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}u\mathrm{d}v,$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{2}$$

Consequently,

$$\begin{aligned}\int_R (x+y)e^{x-y}\,\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_D ue^v\,\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{2}\int_{-1}^{1}\int_{1}^{4} ue^v\,\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{2}\int_{-1}^{1} e^v\frac{u^2}{2}\Big|_1^4\,\mathrm{d}v \\ &= \frac{15}{4}\int_{-1}^{1} e^v\,\mathrm{d}v \\ &= \frac{15}{4}(e^1 - e^{-1}).\end{aligned}$$

□

10. Evaluate $\displaystyle\int_R (x+y)\,\mathrm{d}x\mathrm{d}y$ where R is the rectangle in the xy–plane with vertices $(1,0)$, $(4,3)$, $(3,4)$ and $(0,1)$.

***Solution**:* A sketch of the region R is shown in Figure 9.
We can make the change of variables

$$\begin{aligned} u &= x+y, \\ v &= x-y, \end{aligned}$$

from which we get that

$$\begin{aligned} x &= \frac{1}{2}u + \frac{1}{2}v, \\ y &= \frac{1}{2}u - \frac{1}{2}v. \end{aligned}$$

Then, and (x,y) ranges over the region R, then u and v range over the rectangle defined by $1 \leqslant u \leqslant 7$ and $-1 \leqslant v \leqslant 1$.
By the change of variables formula, we then then have that

$$\int_R (x+y)\,\mathrm{d}x\mathrm{d}y = \int_D u\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}u\mathrm{d}v,$$

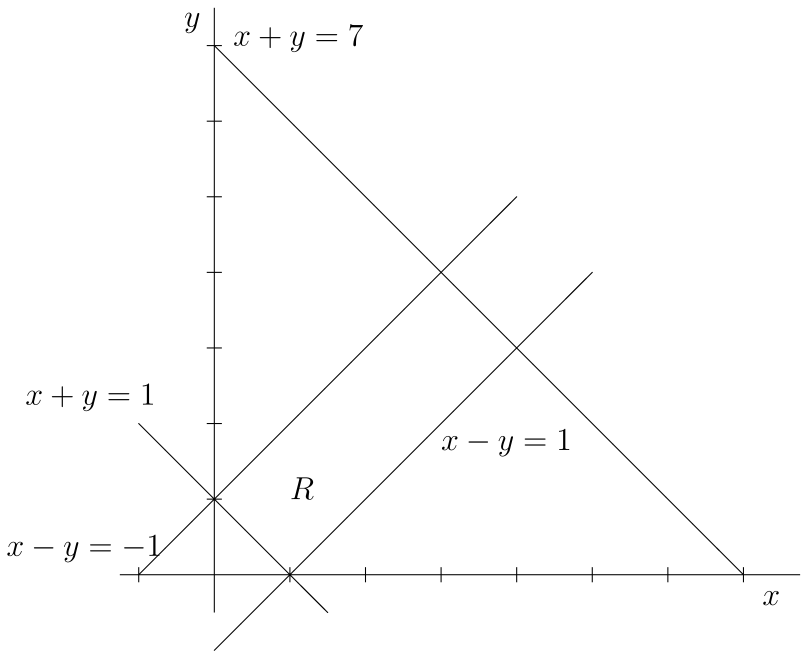

Figure 9: Sketch of Region R in Problem 10

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{2}$$

Consequently,

$$\begin{aligned} \int_R (x+y)\ \mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_D u\ \mathrm{d}u\mathrm{d}v \\ &= \frac{1}{2}\int_{-1}^{1}\int_{1}^{7} u\ \mathrm{d}u\mathrm{d}v \\ &= \frac{1}{2}\int_{-1}^{1} \frac{u^2}{2}\bigg|_1^7\ \mathrm{d}v \\ &= \frac{48}{4}\int_{-1}^{1}\ \mathrm{d}v \\ &= 24. \end{aligned}$$

□

11. Evaluate $\int_R (x-y)\, \mathrm{d}x\mathrm{d}y$ where R is the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$.

***Solution**:* A sketch of R is shown in Figure 10.

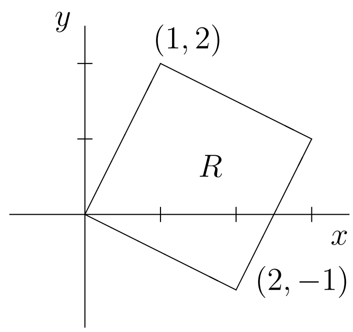

Figure 10: Sketch of Region R in Problem 11

Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear map that takes $(1,0)$ to $(2,-1)$ and $(0,1)$ to $(1,2)$. Then, Φ has the matrix representation

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

When then get the change of variables

$$\begin{aligned} x &= 2u + v, \\ y &= -u + 2v, \end{aligned}$$

which maps the unit square, D, in the uv–plane to R. Consequently, the Change of Variable Formula implies that

$$\int_R (x-y)\, \mathrm{d}x\mathrm{d}y = \int_D (3u - v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v,$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = 5.$$

Evaluating the last integral we obtain

$$\begin{aligned}
\int_R (x-y)\ \mathrm{d}x\mathrm{d}y &= 5\int_0^1\int_0^1 (3u-v)\ \mathrm{d}u\mathrm{d}v \\
&= 5\int_0^1 \left[\frac{3}{2}u^2 - vu\right]_0^1\ \mathrm{d}v \\
&= 5\int_0^1 \left[\frac{3}{2} - v\right]\ \mathrm{d}v \\
&= 5\left[\frac{3}{2}v - \frac{1}{2}v^2\right]_0^1 \\
&= 5.
\end{aligned}$$

□

12. Let $\omega = 2x\ \mathrm{d}x + y\ \mathrm{d}y$ and $\eta = y\ dx - x\ \mathrm{d}y$ denote differential 1–forms. Compute each of the following $\omega\ \mathrm{d}\eta$, $\eta\ \mathrm{d}\omega$ and $\mathrm{d}(\omega\eta)$.

***Solution**:* Compute

$$\mathrm{d}\omega = \mathrm{d}(2x\ \mathrm{d}x + y\ \mathrm{d}y) = 2\mathrm{d}x\mathrm{d}x + \mathrm{d}y\mathrm{d}y = 0,$$

$$\mathrm{d}\eta = \mathrm{d}(y\ dx - x\ \mathrm{d}y) = \mathrm{d}y\mathrm{d}x - \mathrm{d}x\mathrm{d}y = -2\mathrm{d}x\mathrm{d}y.$$

Then

$$\omega\ \mathrm{d}\eta = (2x\ \mathrm{d}x + y\ \mathrm{d}y)(-2\mathrm{d}x\mathrm{d}y) = -4x\mathrm{d}x\mathrm{d}x\mathrm{d}y - 2y\mathrm{d}y\mathrm{d}x\mathrm{d}y = 0,$$

since $\mathrm{d}x\mathrm{d}x = 0$ and $\mathrm{d}y\mathrm{d}x\mathrm{d}y = -\mathrm{d}x\mathrm{d}y\mathrm{d}y = 0$, and

$$\eta\ \mathrm{d}\omega = \eta\cdot 0 = 0.$$

Finally,

$$\begin{aligned}
\omega\eta &= (2x\ \mathrm{d}x + y\ \mathrm{d}y)(y\ dx - x\ \mathrm{d}y) \\
&= 2xy\ \mathrm{d}x\mathrm{d}x - 2x^2\mathrm{d}x\mathrm{d}y + y^2\mathrm{d}y\mathrm{d}x - xy\ \mathrm{d}y\mathrm{d}y \\
&= -(2x^2+y^2)\mathrm{d}x\mathrm{d}y;
\end{aligned}$$

so that

$$\mathrm{d}(\omega\eta) = -(4x\ \mathrm{d}x + 2y\ \mathrm{d}y)\mathrm{d}x\mathrm{d}y = 0.$$

□

13. Let C denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral $\int_C x^3 \, \mathrm{d}y - y^3 \mathrm{d}x$.

    ***Solution**:* Observe that $\int_C x^3 \, \mathrm{d}y - y^3 \mathrm{d}x$ is the flux of the vector field $F(x,y) = x^3 \widehat{\imath} + y^3 \widehat{\jmath}$, so that, by the divergence form of the Fundamental Theorem of Calculus in $\mathbb{R}^2$,

$$\int_C x^3 \, \mathrm{d}y - y^3 \mathrm{d}x = \int_D \mathrm{div} F \, \mathrm{d}x \mathrm{d}y,$$

    where D is the unit disc in $\mathbb{R}^2$ centered at the origin, and

$$\mathrm{div} F = 3x^2 + 3y^2 = 3(x^2 + y^2).$$

    Using polar coordinates we then get that

$$\begin{aligned} \int_C x^3 \, \mathrm{d}y - y^3 \mathrm{d}x &= \int_0^{2\pi} \int_0^1 3r^2 \, r \mathrm{d}r \mathrm{d}\theta \\ &= 6\pi \int_0^1 r^3 \mathrm{d}r \\ &= \frac{3\pi}{2}. \end{aligned}$$

□

14. Let $F(x,y) = y \, \widehat{\imath} - x \, \widehat{\jmath}$ and R be the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\int_{\partial R} F \cdot n \, \mathrm{d}s$.

    ***Solution**:* Observe that the divergence of F is

$$\mathrm{div} F = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0$$

    for all $(x,y) \in \mathbb{R}^2$, so that, by the divergence form of the Fundamental Theorem of Calculus in $\mathbb{R}^2$,

$$\int_{\partial R} F \cdot n \, \mathrm{d}s = \int_R \mathrm{div} F \, \mathrm{d}x \mathrm{d}y = 0.$$

□