Assignment #5

Due on Wednesday, September 23, 2009

Read Section 7.1 on *Limits,* pp. 171–178, in Bressoud.

Background and Definitions

- *(Open Set)* A subset, U, of $\mathbb{R}^n$ is said to be **open** if for any $x \in U$ there exists a positive number r such that $B_r(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$ is entirely contained in U.

  (The empty set, $\emptyset$, is considered to be an open set.)

- *(Continuous Function)* Let U denote an open subset of $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is said to be continuous at $x \in U$ if and only if

$$\lim_{\|y-x\|\to 0} \|F(y) - F(x)\| = 0.$$

Do the following problems

1. Let U_1 and U_2 denote subsets in $\mathbb{R}^n$.

   (a) Show that if U_1 and U_2 are open subsets of $\mathbb{R}^n$, then their intersection

$$U_1 \cap U_2 = \{y \in \mathbb{R}^n \mid y \in U_1 \text{ and } y \in U_2\}$$

   is also open.

   (b) Show that the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 0 \right\}$ is not an open subset of $\mathbb{R}^2$.

2. In Problem 3 of Assignment #3 you proved that every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}$ must be of the form

$$T(v) = w \cdot v \quad \text{for every} \;\; v \in \mathbb{R}^n.$$

   Use this fact together with the Cauchy–Schwarz inequality to prove that T is continuous at every point in $\mathbb{R}^n$.

3. A subset, U, of $\mathbb{R}^n$ is said to be **convex** if given any two points x and y in U, the straight line segment connecting them is entirely contained in U; in symbols,

$$\{x + t(y - x) \in \mathbb{R}^n \mid 0 \leq t \leqslant 1\} \subseteq U$$

   (a) Prove that the ball $B_r(O) = \{x \in \mathbb{R}^n \mid \|x\| < R\}$ is a convex subset of $\mathbb{R}^n$.

   (b) Prove that the "punctured unit disc" in $\mathbb{R}^2$,

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; 0 < x^2 + y^2 < 1 \right\},$$

   is not a convex set.

4. Let x and y denote real numbers.

   (a) Starting with the self–evident inequality: $(|x| - |y|)^2 \geqslant 0$, derive the inequality

$$|xy| \leqslant \frac{1}{2}(x^2 + y^2).$$

   (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

   Use the inequality derived in the previous part to prove that f is continuous at the origin.

5. Exercise 10 on page 180 in the text.