Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by
$$4x - y - 3z = 12.$$

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\begin{cases} x = -1 + 4t, \\ y = -7t, \\ z = 2 - t. \end{cases}$$

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1, 1, 0)$, $(2, 0, 1)$ and $(0, 3, 1)$

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map
$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n$$
is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

6. Let $U \subseteq \mathbb{R}^n$ be open and $F\colon U \to \mathbb{R}^m$ be function satisfying
$$\|F(v) - F(w)\| \leqslant K\|v - w\|^\alpha \quad \text{for all } v, w \in U,$$
and some positive constants K and α.

   Prove that F is continuous on U.

7. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is continuous at $(0,0)$.

8. Show that

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$.

9. Determine the value of L that would make the function

$$f(x,y) = \begin{cases} x \sin\left(\dfrac{1}{y}\right) & \text{if } y \neq 0; \\ \\ L & \text{otherwise }, \end{cases}$$

continuous at $(0,0)$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous on $\mathbb{R}^2$? Justify your answer.

10. Define $G\colon \mathbb{R}^2 \to \mathbb{R}$ by $G(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$. Prove that G is continuous on $\mathbb{R}^2$; that is, prove that

$$\lim_{(x,y)\to(x_o,y_o)} G(x,y) = G(x_o,y_o) \quad \text{for all } (x_o,y_o) \in \mathbb{R}^2$$

or

$$\lim_{(x,y)\to(x_o,y_o)} |G(x,y) - G(x_o,y_o)| = 0 \quad \text{for all } (x_o,y_o) \in \mathbb{R}^2$$

11. Let U denote an open subset of $\mathbb{R}^2$ and let $g\colon U \to \mathbb{R}$ be two scalar fields on U. Assume that $g(x_o,y_o) \neq 0$ for some $(x_o,y_o) \in U$. Prove that if g is continuous at (x_o,y_o), then there exists $\delta > 0$ such that $B_\delta(x_o,y_o) \subseteq U$ and

$$(x,y) \in B_\delta(x_o,y_o) \Rightarrow |g(x,y)| > \frac{|g(x_o,y_o)|}{2}.$$

Suggestion: Consider $\varepsilon = \dfrac{|g(x_o, y_o)|}{2} > 0$.

12. Let U, g and (x_o, y_o) be as in the previous problem. Assume that $g(x_o, y_o) \neq 0$ and that g is continuous at (x_o, y_o). Put

$$h(x, y) = \frac{1}{g(x, y)}.$$

Prove that h is continuous at (x_o, y_o).

Suggestion: Use the result of the previous problem and the Squeeze Theorem.