Exam 2

October 28, 2009

Name: ____________________________

This is a closed book exam. Show all significant work and justify all your answers. Use your own paper and/or the paper provided by the instructor. You have 75 minutes to work on the following 6 problems. Relax.

1. Let U denote an open subset of $\mathbb{R}^n$, and let $F\colon U \to \mathbb{R}^m$ be a vector field on U.
   (a) State precisely what it means for F to be differentiable at $u \in U$.
   (b) Suppose that a vector field $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Prove that F is differentiable at every $u \in U$, and compute its derivative map,
   $$DF(u)\colon \mathbb{R}^n \to \mathbb{R}^m,$$
   at u, for all $u \in \mathbb{R}^n$.

2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ denote the scalar field on $\mathbb{R}^2$ defined by
   $$f(x, y) = x^{2/3}y^{1/3} \quad \text{for all} \quad (x, y) \in \mathbb{R}^2.$$
   (a) Show that the partial derivatives of f at $(0, 0)$ exist and compute them.
   (b) Show that f is not differentiable at $(0, 0)$.

3. For fixed vectors v and u in $\mathbb{R}^n$, define $\sigma\colon \mathbb{R} \to \mathbb{R}^n$ by
   $$\sigma(t) = u + tv \quad \text{for all } t \in \mathbb{R}.$$
   Prove that σ is differentiable at every $t \in \mathbb{R}$ and compute its derivative map, $D\sigma(t)$, for all $t \in \mathbb{R}$

4. Let I denote an open interval of real numbers and define $\sigma\colon\ I \to \mathbb{R}^2$ by
   $$\sigma(t) = (x(t), y(t)) \quad \text{for all } t \in I,$$
   where $x(t)$ and $y(t)$ are real valued functions of $t \in I$.

   Prove that σ is differentiable at $t \in I$ if and only if both x and y are differentiable at $t \in I$. Furthermore, $D\sigma(t)h = h(x'(t), y'(t))$ for all $h \in \mathbb{R}$.

5. Let U denote an open subset of $\mathbb{R}^n$ and Q an open subset of $\mathbb{R}^m$. Consider the maps $F\colon U \to \mathbb{R}^m$ and $G\colon Q \to \mathbb{R}^k$.

   (a) State the Chain Rule in the context of the functions F and G and the open sets given above. Be explicit as to what your assumptions and conclusions are.

   (b) Use the Chain Rule to prove the following: If f is a differentiable scalar field on an open set $U \in \mathbb{R}^n$ and $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path such that $\sigma(I) \subseteq U$, then the function $g\colon I \to \mathbb{R}$ defined by

   $$g(t) = f(\sigma(t)) \quad \text{for all } t \in I,$$

   is differentiable on I. Give a formula for computing $g'(t)$ for all $t \in I$ in terms of the gradient of f and $\sigma'(t)$.

   (c) Use your result from the previous part to prove that, if f is differentiable at $u \in U$, then the limit

   $$\lim_{t\to 0} \frac{f(u + t\widehat{v}) - f(u)}{t},$$

   where $\widehat{v}$ denotes a unit vector, exists. Give an interpretation to your result.

   (d) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 3xy - y^2$ for all $(x, y) \in \mathbb{R}^2$. Compute the gradient of f at the point $(2, 1)$, and find a direction, $\widehat{v}$, along which f is increasing the fastest at $(2, 1)$. Justify your result.

6. Let f be a scalar field in $\mathbb{R}^n$ defined by $f(v) = \|v\|^2$ for all $v \in \mathbb{R}^n$.

   (a) Prove that f is differentiable on $\mathbb{R}^n$ and use this fact to prove that the function $g\colon I \to \mathbb{R}$ defined by

   $$g(t) = \|\sigma(t)\|^2, \quad \text{for all } \ t \in I,$$

   where the path σ is differentiable on an open interval I, is differentiable and compute $g'(t)$.

   (b) Let g be the function defined in part (a) above. Prove that if g has a critical point at $t_o \in I$, then $\sigma(t_o)$ and $\sigma'(t_o)$ are orthogonal (or perpendicular) to each other.

   (c) Find the point (or points) along the path in $\mathbb{R}^2$ given by

   $$\sigma(t) = (t, t^2 - 1), \quad \text{for } t \in \mathbb{R},$$

   which are the closest to the origin $(0, 0)$ in $\mathbb{R}^2$.