Review Problems for Exam 2

1. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \dfrac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in \mathbb{R}^n$. What is the gradient of f at u for all $x \in \mathbb{R}^n$?

2. Let $g\colon\ [0,\infty) \to \mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x,y) = g(r)$ where $r = \sqrt{x^2+y^2}$.

   (a) Compute $\dfrac{\partial r}{\partial x}$ in terms of x and r, and $\dfrac{\partial r}{\partial y}$ in terms of y and r.

   (b) Compute ∇f in terms of $g'(r)$, r and the vector $\mathbf{r} = x\widehat{\imath} + y\widehat{\jmath}$.

3. Let $f\colon U \to \mathbb{R}$ denote a scalar field defined on an open subset U of $\mathbb{R}^n$, and let $\widehat{u}$ be a unit vector in $\mathbb{R}^n$. If the limit

$$\lim_{t\to 0} \frac{f(v+t\widehat{u}) - f(v)}{t}$$

   exists, we call it the *directional derivative of f at v in the direction of the unit vector $\widehat{u}$*. We denote it by $D_{\widehat{u}}f(v)$.

   (a) Show that if f is differentiable at $v \in U$, then, for any unit vector $\widehat{u}$ in $\mathbb{R}^n$, the directional derivative of f in the direction of $\widehat{u}$ at v exists, and

$$D_{\widehat{u}}f(v) = \nabla f(v) \cdot \widehat{u},$$

   where $\nabla f(v)$ is the gradient of f at v.

   (b) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Prove that if $D_{\widehat{u}}f(v) = 0$ for every unit vector $\widehat{u}$ in $\mathbb{R}^n$, then $\nabla f(v)$ must be the zero vector.

   (c) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Use the Cauchy–Schwarz inequality to show that the largest value of $D_{\widehat{u}}f(v)$ is $\|\nabla f(v)\|$ and it occurs when $\widehat{u}$ is in the direction of $\nabla f(v)$.

4. The scalar field $f\colon U \to \mathbb{R}$ is said to have a *local minimum* at $x \in U$ if there exists $r > 0$ such that $B_r(x) \subseteq U$ and

$$f(x) \leqslant f(y) \quad \text{for every} \quad y \in B_r(x).$$

   Prove that if f is differentiable at $x \in U$ and f has a local minimum at x, then $\nabla f(x) = \mathbf{0}$, the zero vector in $\mathbb{R}^n$.

5. Let I denote an open interval in $\mathbb{R}$. Suppose that $\sigma\colon I \to \mathbb{R}^n$ and $\gamma\colon I \to \mathbb{R}^n$ are paths in $\mathbb{R}^n$. Define a real valued function $f\colon I \to \mathbb{R}$ of a single variable by

$$f(t) = \sigma(t) \cdot \gamma(t) \quad \text{for all} \quad t \in I;$$

that is, $f(t)$ is the dot product of the two paths at t.

Show that if σ and γ are both differentiable on I, then so is f, and

$$f'(t) = \sigma'(t) \cdot \gamma(t) + \sigma(t) \cdot \gamma'(t) \quad \text{for all} \quad t \in I.$$

6. Let $\sigma\colon I \to \mathbb{R}^n$ denote a differentiable path in $\mathbb{R}^n$. Show that if $\|\sigma(t)\|$ is constant for all $t \in I$, then $\sigma'(t)$ is orthogonal to $\sigma(t)$ for all $t \in I$.

7. A particle is following a path in three–dimensional space given by

$$\sigma(t) = (e^t, e^{-t}, 1 - t) \quad \text{for} \quad t \in \mathbb{R}.$$

At time $t_o = 1$, the particle flies off on a tangent.

   (a) Where will the particle be at time $t_1 = 2$?

   (b) Will the particle ever hit the xy–plane? Is so, find the location on the xy plane where the particle hits.

8. Let U denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at every $x \in U$. Fix x and y in U, and define $g\colon [0, 1] \to \mathbb{R}$ by

$$g(t) = f(x + t(y - x)) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

   (a) Explain why the function g is well defined.

   (b) Show that g is differentiable on $(0, 1)$ and that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for} \quad 0 < t < 1.$$

(*Suggestion:* Consider

$$\frac{g(t + h) - g(t)}{h} = \frac{f(x + t(y - x) + h(y - x)) - f(x + t(y - x))}{h}$$

and apply the definition of differentiability of f at the point $x + t(y - x)$.)

(c) Use the Mean Value Theorem for derivatives to show that there exists a point z is the line segment connecting x to y such that

$$f(y) - f(x) = D_{\widehat{u}} f(z) \|y - x\|,$$

where $\widehat{u}$ is the unit vector in the direction of the vector $y - x$; that is, $\widehat{u} = \dfrac{1}{\|y - x\|}(y - x)$.
(*Hint:* Observe that $g(1) - g(0) = f(y) - f(x)$.)

9. Prove that if U is an open and convex subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is differentiable on U with $\nabla f(v) = \mathbf{0}$ for all $v \in U$, then f must be a constant function.

10. Let f be a scalar field defined on (x, y) where $x = r\cos\theta$, $y = r\sin\theta$. Show that
$$\nabla f = \frac{\partial f}{\partial r}\overrightarrow{\mathbf{u}_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\overrightarrow{\mathbf{u}_\theta},$$
where $\overrightarrow{\mathbf{u}_r} = (\cos\theta, \sin\theta)$ and $\overrightarrow{\mathbf{u}_\theta} = (-\sin\theta, \cos\theta)$.

    Hint: First find $\partial f/\partial r$ and $\partial f/\partial \theta$ in terms of $\partial f/\partial x$ and $\partial f/\partial y$ and then solve for $\partial f/\partial x$ and $\partial f/\partial y$ int terms of $\partial f/\partial r$ and $\partial f/\partial \theta$.

11. Let U be an open subset of $\mathbb{R}^n$ and I be an open interval. Suppose that $f\colon U \to \mathbb{R}$ is a differentiable scalar field and $\sigma\colon I \to \mathbb{R}^n$ be a differentiable path whose image lies in U. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if f has a local maximum or a local minimum at some point on the path, then ∇f is perpendicular to the path at that point.

    Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

12. Let $\sigma\colon [a, b] \to \mathbb{R}^n$ be a differentiable, one–to–one path. Suppose also that $\sigma'(t)$, is never the zero vector. Let $h\colon [c, d] \to [a, b]$ be a one–to–one and onto map such that $h'(t) \neq 0$ for all $t \in [c, d]$. Define
$$\gamma(t) = \sigma(h(t)) \quad \text{for all} \quad t \in [c, d].$$
$\gamma\colon [c, d] \to \mathbb{R}^n$ is a called a *reparametrization* of σ

    (a) Show that γ is a differentiable, one–to–one path.
    (b) Compute $\gamma'(t)$ and show that it is never the zero vector.
    (c) Show that σ and γ have the same image in $\mathbb{R}^n$.