Review Problems for Exam 3

1. Consider a wheel of radius a which is rolling on the x–axis in the xy–plane. Suppose that the center of the wheel moves in the positive x–direction and a constant speed v_o. Let P denote a fixed point on the rim of the wheel.

   (a) Give a path $\sigma(t) = (x(t), y(t))$ giving the position of the P at any time t, if P is initially at the point $(0, 2a)$.

   (b) Compute the velocity of P at any time t. When is the velocity of P horizontal? What is the speed of P at those times?

2. Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$; i.e., C is the upper unit semi–circle. C can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1-\tau^2}) \quad \text{for} \quad -1 \leqslant \tau \leqslant 1.$$

   (a) Compute $s(t)$, the arclength along C from $(-1, 0)$ to the point $\sigma(t)$, for $0 \leqslant t \leqslant 1$.

   (b) Compute $s'(t)$ for $-1 < t < t$ and sketch the graph of s as function of t.

   (c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leqslant t \leqslant 1$, and deduce that

$$\sin(s(t)) = \sqrt{1-t^2} \quad \text{for all} \quad -1 \leqslant t \leqslant 1.$$

3. Let C denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral $\int_C \frac{x}{2}\ \mathrm{d}y - \frac{y}{2}\ \mathrm{d}x$.

4. Let $F(x, y) = 2x\ \widehat{i} - y\ \widehat{j}$ and R be the square in the xy–plane with vertices $(0, 0)$, $(2, -1)$, $(3, 1)$ and $(1, 2)$. Evaluate $\oint_{\partial R} F \cdot n\ \ \mathrm{d}s$.

5. Evaluate the line integral $\int_{\partial R} (x^4 + y)\ \ \mathrm{d}x + (2x - y^4)\ \ \mathrm{d}y$, where R is the rectangular region

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3,\ -2 \leqslant y \leqslant 1\},$$

   and ∂R is traversed in the counterclockwise sense.

6. Integrate the function given by $f(x, y) = xy^2$ over the region, R, defined by:

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, 0 \leqslant y \leqslant 4 - x^2\}.$$

7. Let R denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

for $a > 0$ and $b > 0$.

(a) Evaluate the line integral $\oint_{\partial R} x \, \mathrm{d}y - y \, \mathrm{d}x$, where ∂R is the ellipse in (1) traversed in the positive sense.

(b) Use your result from part (a) and the divergence form of Green's theorem to come up with a formula for computing the area of the region enclosed by the ellipse in (1).

8. Evaluate the double integral $\int_R e^{-x^2} \, \mathrm{d}x \, \mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 1.

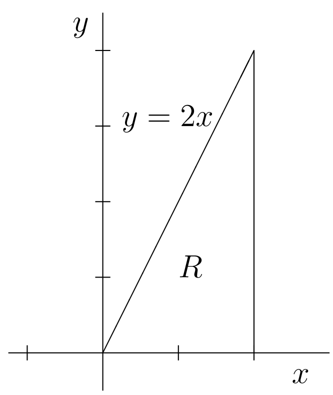

Figure 1: Sketch of Region R in Problem 8