Assignment #1

Due on Monday September 7, 2009

Read Section 5.1 on *Sampling and Statistics,* pp. 233–236, in Hogg, Craig and McKean.

Background and Definitions

1. **Moment Generating Function.** Given a random variable, X, its moment generating function (mgf), M_X, is given by

$$M_X(t) = E(e^{tX}),$$

for values of t in some interval around 0, where $E(Y)$ denotes the expected value of a random variable Y. Thus, if X is continuous with probability density function (pdf) f_X,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(t)\, \mathrm{d}x;$$

if X is discrete with probability mass function (pmf), p_X,

$$M_X(t) = \sum_k e^{tk} p_X(k).$$

For example, if $Z \sim \text{normal}(0,1)$ then

$$M_Z(t) = e^{t^2/2} \quad \text{for all } t \in \mathbb{R};$$

and if $X \sim \text{binomial}(n,p)$, then

$$M_X(t) = (1 - p + pe^t)^n \quad \text{for all } t \in \mathbb{R}.$$

The importance of the mgf is that it is uniquely determined by the underlying distribution and, conversely, it uniquely determines the underlying distribution. Thus, for instance, if we can show that $M_Y(t) = e^{t^2/2}$ for all $t \in \mathbb{R}$, then Y must follow a standard normal distribution.

Finally, if $X_1, X_2, \ldots, X_n$ are independent random variables with mgfs

$$M_{X_1}, M_{X_2}, \ldots, M_{X_n},$$

respectively, defined on some interval around 0, then

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$

for t in the common interval of definition.

2. **The Gamma Function.** The gamma function, $\Gamma(x)$, plays a very important role in the definitions a several probability distributions which are very useful in statistical inference. It is defined as follows

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t \quad \text{for all} \quad x > 0. \tag{1}$$

Note: $\Gamma(x)$ can also be defined for negative values of x which are not integers; it is not defined at $x = 0$. In this course, we will only consider $\Gamma(x)$ for $x > 0$.

In problems 3–5 you will derive a few important properties of this function.

Do the following problems

1. Let $0 < p < 1$. A random variable X is said to follow a Bernoulli(p) distribution if X takes the values 0 and 1, $p_X(0) = 1 - p$ and $p_X(1) = p$.

   Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Bernoulli(p) distribution and define the statistic $Y = X_1 + X_2 + \cdots + X_n$.

   (a) Compute the mgf of Y and use it to determine the sampling distribution of Y.

   (b) Show that Y/n is an unbiased estimator of p.

2. A random variable, X, is said to follow an exponential distribution with parameter β, where $\beta > 0$, if X has the pdf

$$f_X(x) = \begin{cases} \dfrac{1}{\beta} e^{-x/\beta} & \text{if } \ x > 0; \\ \\ 0 & \text{if } \ x \leqslant 0. \end{cases}$$

   We write $X \sim \text{exponential}(\beta)$.

   (a) Let $\beta > 0$ and $X \sim \text{exponential}(\beta)$. Verify that the mgf of X is

$$M_X(t) = \frac{1}{1 - \beta t} \quad \text{for} \quad t < \frac{1}{\beta}.$$

   (b) Let $\beta > 0$ and $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β) distribution. Compute the mgf of the sample mean, $\overline{X}_n$.

   (c) Let $Y_n = 2n\overline{X}_n/\beta$. Compute the mgf of Y_n.

3. Let $\Gamma\colon (0,\infty) \to \mathbb{R}$ be as defined in (1). Derive the following identities:

   (a) $\Gamma(1) = 1$.

   (b) $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.

   (c) $\Gamma(n+1) = n!$ for all positive integers n.

4. Let $\Gamma\colon (0,\infty) \to \mathbb{R}$ be as defined in (1).

   (a) Compute $\Gamma(1/2)$.
   Hint: The change of variable $t = z^2/2$ might come in handy. Recall that if $Z \sim \text{normal}(0,1)$, then its pdf is given by

   $$f_Z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \quad \text{for all } z \in \mathbb{R}.$$

   (b) Compute $\Gamma(3/2)$.

5. Use the results of Problems 3 and 4 to derive the identity:

   $$\Gamma\left(\frac{k}{2}\right) = \frac{\Gamma(k)\sqrt{\pi}}{2^{k-1}\,\Gamma\left(\frac{k+1}{2}\right)}$$

   for every positive, odd integer k.

   Suggestion: Proceed by induction.