Solutions to Assignment #10

1. We wish to determine whether a given coin is fair or not. Thus, we test the null hypothesis H_o: $p = \dfrac{1}{2}$ versus the alternative hypothesis H_1: $p \neq \dfrac{1}{2}$. An experiment consisting of 10 independent flips of the coin and observing the number of heads, Y, in the 10 trials. The statistic Y is the test statistic for the test. The following rejection criterion is set: Reject H_o is either $Y = 0$ or $Y = 10$.

   (a) What is the significance level of the test?

   Solution: The significance level, α, of a test is the largest probability of rejecting the null hypothesis when it is in fact true. In this case, if H_o is true, the test statistic, Y, has binomial$(10, 0.5)$ distribution. Thus, the probability of rejecting the null hypothesis if it is true is

   $$\alpha = \mathrm{P}(Y = 0) + \mathrm{P}(Y = 10) = \left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10} = \frac{1}{2^9},$$

   or

   $$\alpha = \frac{1}{512}.$$
   $\square$

   (b) If the coin is in fact loaded, with the probability of a head being $3/4$, what is the power of the test?

   Solution: The power of a test is the probability of rejecting the null hypothesis when it is false. In this case it is

   $$\gamma = \mathrm{P}(Y = 0) + \mathrm{P}(Y = 10),$$

   where $Y \sim$ binomial$(10, 3/4)$. Thus,

   $$\gamma = \left(1 - \frac{3}{4}\right)^{10} + \left(\frac{3}{4}\right)^{10} = \frac{1 + 3^{10}}{4^{10}} \approx 0.0563.$$
   $\square$

2. Suppose that $Y \sim$ binomial$(100, p)$ consider a test that rejects H_o : $p = 0.5$ in favor of the alternative hypothesis H_1 : $p \neq 0.5$ provided that $|Y - 50| > 10$. Use the Central Limit Theorem to answer the following questions:

(a) Determine α for this test.

> **Solution**: The significance level of the test, α, is the largest probability that the the tests rejects the null hypothesis when it is in fact true. In this case, the null hypothesis consists of the single value 0.5. It then follows that
>
> $$\alpha = \mathrm{P}(|Y - 50| > 10),$$
>
> where $Y \sim$ binomial$(100, 0.5)$. We may use the Central Limit Theorem to approximate this value as follows.
>
> $$\alpha = 1 - \mathrm{P}\left(|Y - 50| \leqslant 10\right)$$
>
> $$= 1 - \mathrm{P}\left(40 \leqslant Y \leqslant 60\right)$$
>
> $$= 1 - \mathrm{P}\left(39.5 < Y \leqslant 60.5\right),$$
>
> where we have used the continuity correction, since we are going to use the Central Limit Theorem to approximate a discrete distribution; namely, Y has an approximate normal$(50, 25)$ distribution in this case. We then have that
>
> $$\alpha \approx 1 - \mathrm{P}\left(39.5 < W \leqslant 60.5\right),$$
>
> where $W \sim$ normal$(50, 25)$. Consequently,
>
> $$\alpha \approx 1 - \left(F_W(60.5) - F_W(39.5)\right),$$
>
> where F_W is cdf of $W \sim$ normal$(50, 25)$. We then have that
>
> $$\alpha \approx 0.0357.$$
>
> $\square$

(b) Estimate the power of the test as a function of p and graph it.

> **Solution**: The power of a test is the probability that null hypothesis will be rejected when it is false. In this case, H$_o$ is false when $p \neq 0.5$. Thus, for each $p \neq 0.5$, the power of the test is a function of p given by
>
> $$\gamma(p) = \mathrm{P}(|Y - 50| > 10),$$
>
> where
>
> $$Y \sim \text{binomial}(100, p) \quad \text{for} \quad p \neq \frac{1}{2}.$$

We then have that

$$
\begin{aligned}
\gamma(p) &= \mathrm{P}(|Y - 50| > 10) \\[2mm]
&= 1 - \mathrm{P}(|Y - 50| \leqslant 10) \\[2mm]
&= 1 - \mathrm{P}(40 \leqslant T \leqslant 60) \\[2mm]
&= 1 - \mathrm{P}(39.5 < T \leqslant 60.5),
\end{aligned}
$$

where we have used again the continuity correction, since we are going to be applying the Central Limit Theorem to approximate the distribution of Y by that of a normal$(100p, 100p(1-p))$ random variable. We then have that

$$
\gamma(p) \approx 1 - \mathrm{P}(39.5 < W_p \leqslant 60.5),
$$

where W_p denotes a normal$(100p, 100p(1-p))$ random variable. Thus,

$$
\gamma(p) \approx 1 - (F_{W_p}(60.5) - F_{W_p}(39.5)), \tag{1}
$$

where F_{W_p} denotes the cdf of $W_p \sim$ normal$(100p, 100p(1-p))$. The formula in (1) can be input into R by typing the following commands:

```
p <- seq(0.01,0.99, by=0.01)
```

which sets up and array of values of p from 0.01 to 0.99 in steps of $1/100$,

```
gammap <- 1-(pnorm(60.5,100*p,sqrt(100*p*(1-p)))
                -pnorm(39.5,100*p,sqrt(100*p*(1-p))))
```

which computes the approximate values of $\gamma(p)$ according to formula (1), and stores then in an array called `gammap`. We can then use the `plot` function in R by typing

```
plot(p,gammap,type='l',ylab="Power at p")
```

The last R command generates a sketch of the graph of $\gamma(p)$ as a function of p shown in Figure 1 on page 4. □
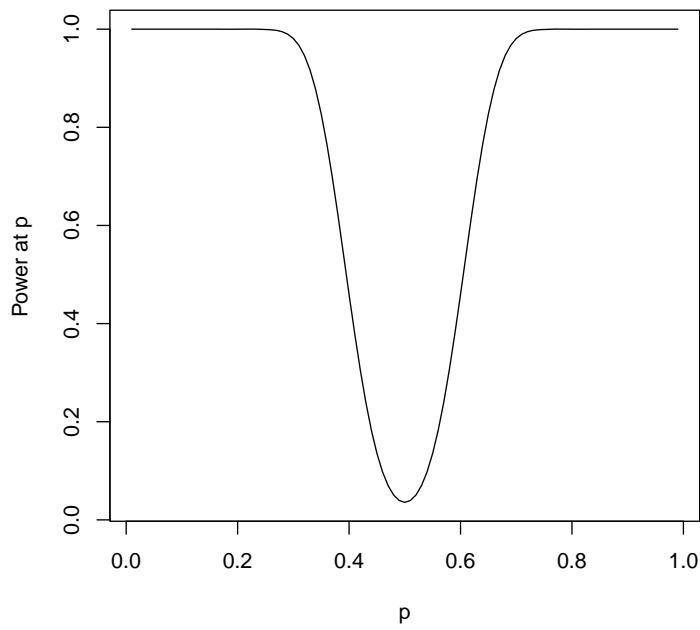
Figure 1: Sketch of graph of power function for test in Problem 2

3. Let $X_1, X_2, \ldots, X_8$ denote a random sample of size $n = 8$ from a Poissonλ distribution and put $Y = \sum\limits_{j=1}^{8} X_j$. Reject the null hypothesis H$_o$: $\lambda = 0.5$ in favor of the alternative H$_1$: $\lambda > 0.5$ provided that $Y \geqslant 8$.

(a) Compute the significance level, α, of the test.

Solution: The significance level, α, is the largest probability that the test will reject H$_o$ when it is in fact true. Since, H$_o$ is true only when $\lambda = 0.5$, we get that

$$\alpha = \mathrm{P}(Y \geqslant 8),$$

where $Y \sim \text{Poisson}(8(0.5))$, since Y is the sum of 8 independent Poisson(0.5) random variables if H$_o$ is true. It then follows that

$$\alpha = \sum_{k=8}^{\infty} \frac{4^k}{k!} e^{-4}$$

$$= 1 - e^{-4}\left(1 + 4 + \frac{4^2}{2} + \frac{4^3}{3!} + \cdots + \frac{4^7}{7!}\right)$$

$$\approx 0.0511.$$

$\square$

(b) Determine the power function, γ, as a function of λ.

Solution: The power of the test is the probability of the test will reject H$_o$ when it is false. Thus,

$$\gamma(\lambda) = \mathrm{P}(Y \geqslant 8),$$

where $Y \sim \text{Poisson}(8\lambda)$, where $\lambda \neq 0.5$. Thus,

$$\gamma(\lambda) = 1 - e^{-8\lambda}\left(1 + 8\lambda + \frac{(8\lambda)^2}{2} + \frac{(8\lambda)^3}{3!} + \cdots + \frac{(8\lambda)^7}{7!}\right).$$

$\square$

4. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size $n = 25$ from a Normal$(\mu, 100)$. Reject the null hypothesis H$_o$: $\mu = 0$ in favor of the alternative H$_1$: $\mu = 1.5$ provided that $\overline{X}_n > 0.64$.

(a) Compute the significance level, α, of the test.

> **Solution:** The significance level, α, is the probability that the test will reject the null hypothesis when in fact H_o is true. Thus,
>
> $$\alpha = \mathrm{P}(\overline{X}_n > 0.64),$$
>
> where $\overline{X}_n \sim \mathrm{normal}(0, 100/25)$, or $\overline{X}_n \sim \mathrm{normal}(0, 4)$. We then have that
>
> $$\alpha = 1 - \mathrm{P}(\overline{X}_n \leqslant 0.64) = 1 - F_{\overline{X}_n}(0.64),$$
>
> where $\overline{X}_n \sim \mathrm{normal}(0, 4)$; thus, $\alpha \approx 0.3745$.   □

(b) What is the power of the test?

> **Solution:** The power of the test is the probability that the test will reject the null hypothesis when it is false. In this case, the power, γ, is
>
> $$\gamma = \mathrm{P}(\overline{X}_n > 0.64),$$
>
> where $\overline{X}_n \sim \mathrm{normal}(1.5, 4)$, or
>
> $$\gamma = 1 - \mathrm{P}(\overline{X}_n \leqslant 0.64) = 1 - F_{\overline{X}_n}(0.64),$$
>
> where $\overline{X}_n \sim \mathrm{normal}(1.5, 4)$; thus, $\gamma \approx 0.6664$.   □

5. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a $\mathrm{Normal}(\mu, \sigma^2)$. Consider testing the null hypothesis

$$\mathrm{H}_o\colon \ \mu = \mu_o \tag{2}$$

against the alternative $\mathrm{H}_1\colon \ \mu \neq \mu_o$. Suppose that the test rejects H_o provided that $|\overline{X}_n - \mu_o| > b$ for some $b > 0$.

(a) If the significance level of the test is α, determine the value of b. *Suggestion:* Use the $t(n-1)$ distribution.

> **Solution:** Consider the T_n statistic defined by
>
> $$T_n = \frac{\overline{X}_n - \mu_o}{S_n/\sqrt{n}},$$
>
> where S_n^2 is the sample variance. Then, the rejection region, R, is defined by the condition
>
> $$|T_n| > \frac{\sqrt{n}}{S_n}b.$$

Now, if the null hypothesis is true, it follows that $T_n \sim t(n-1)$, and

$$\mathrm{P}\left(|T_n| > \frac{\sqrt{n}}{S_n}b\right) = \alpha.$$

We therefore want

$$\frac{\sqrt{n}}{S_n}b = t_{\alpha/2,n-1},$$

where $t_{\alpha/2,n-1}$ has the property that

$$\mathrm{P}(|T_n| > t_{\alpha/2,n-1}) = \alpha, \quad \text{where } T_n \sim t(n-1).$$

We then have that

$$b = t_{\alpha/2,n-1}\,\frac{S_n}{\sqrt{n}}.$$

$\square$

(b) Using the value of b found in part (b), obtain a range of values, depending on the sample mean and standard deviation, $\overline{X}_n$ and S_n, respectively, for μ_o such that the null hypothesis, H_o, in (2) is not rejected.

Solution: The null hypothesis, $\mathrm{H}_o\colon \mu = \mu_o$, is not rejected if

$$|\overline{X}_n - \mu_o| \leqslant t_{\alpha/2,n-1}\,\frac{S_n}{\sqrt{n}},$$

or

$$|\mu_o - \overline{X}_n| \leqslant t_{\alpha/2,n-1}\,\frac{S_n}{\sqrt{n}}.$$

Hence, $\mathrm{H}_o\colon \mu = \mu_o$ is not rejected if

$$\overline{X}_n - t_{\alpha/2,n-1}\,\frac{S_n}{\sqrt{n}} \leqslant \mu_o \leqslant \overline{X}_n + t_{\alpha/2,n-1}\,\frac{S_n}{\sqrt{n}}.$$

In other words, $\mathrm{H}_o\colon \mu = \mu_o$ is not rejected if μ_o lies in the interval

$$\left[\overline{X}_n - t_{\alpha/2,n-1}\,\frac{S_n}{\sqrt{n}}, \overline{X}_n + t_{\alpha/2,n-1}\,\frac{S_n}{\sqrt{n}}\right],$$

which is a $100(1-\alpha)\%$ confidence interval for the mean, μ_o, of the distribution. $\square$