Assignment #11

Due on Monday, November 9, 2009

Read Section 6.3 on *Maximum Likelihood Tests,* pp. 333–339, in Hogg, Craig and McKean.

Background and Definitions

Likelihood Functions and Likelihood Ratio Tests

- **Likelihood Function.** Given a random sample, $X_1, X_2, \ldots, X_n$, from a distribution with distribution function $f(x \mid \theta)$, either a pdf or a pmf, where θ is some unknown parameter (either a scalar or a vector parameter), the joint distribution the sample is given by

  $$f(x_1, x_2, \ldots, x_n \mid \theta) = f(x_1 \mid \theta) \cdot f(x_2 \mid \theta) \cdots f(x_n \mid \theta),$$

  by the independence condition in the definition of a random sample. If the random variables, $X_1, X_2, \ldots, X_n$, are discrete, $f(x_1, x_2, \ldots, x_n \mid \theta)$ gives the probability of observing the values

  $$X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n,$$

  under the assumption that the sample is taken from certain distribution with parameter θ. We can also interpret $f(x_1, x_2, \ldots, x_n \mid \theta)$ as measuring the likelihood that the parameter will be θ given that we have observed the values $x_1, x_2, \ldots, x_n$ in the sample. Thus, we call $f(x_1, x_2, \ldots, x_n \mid \theta)$ the **likelihood function** for the parameter θ given the observations $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and denote it by $L(\theta \mid x_1, x_2, \ldots, x_n)$; that is,

  $$L(\theta \mid x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n \mid \theta).$$

- **Likelihood Ratio Test.** For a hypothesis test of

  $$\mathrm{H}_o\colon \theta \in \Omega_o$$

  against the alternative

  $$\mathrm{H}_1\colon \theta \in \Omega_1,$$

  based on a random sample, $X_1, X_2, \ldots, X_n$, from a distribution with function $f(x \mid \theta)$, the **likelihood ratio statistic**, $\Lambda(x_1, x_2, \ldots, x_n)$, is defined by

  $$\Lambda(x_1, x_2, \ldots, x_n) = \frac{\displaystyle\sup_{\theta \in \Omega_o} L(\theta \mid x_1, x_2, \ldots, x_n)}{\displaystyle\sup_{\theta \in \Omega} L(\theta \mid x_1, x_2, \ldots, x_n)},$$

where $\Omega = \Omega_o \cup \Omega_1$ with $\Omega_o \cap \Omega_1 = \emptyset$. We can then define the rejection region

$$R = \{(x_1, x_2, \ldots, x_n) \mid \Lambda(x_1, x_2, \ldots, x_n) \leqslant c\},$$

for some critical value c with $0 < c < 1$. This defines a **likelihood ratio test** for H_o against H_1.

- **Maximum Likelihood Estimator.** A value, $\widehat{\theta}$, for the parameter θ such that

$$L(\widehat{\theta} \mid x_1, x_2, \ldots, x_n) = \sup_{\theta \in \Omega} L(\theta \mid x_1, x_2, \ldots, x_n)$$

is called a **maximum likelihood estimator** for θ, or an MLE for θ. Thus, if $\widehat{\theta}$ is an MLE for θ, then

$$\Lambda(x_1, x_2, \ldots, x_n) = \frac{L(\theta_o \mid x_1, x_2, \ldots, x_n)}{L(\widehat{\theta} \mid x_1, x_2, \ldots, x_n)}$$

is the Likelihood ratio statistic for the test of $\mathrm{H}_o\colon\ \theta = \theta_o$ versus the alternative $\mathrm{H}_1\colon\ \theta \neq \theta_o$.

Do the following problems

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β), for $\beta > 0$.

   (a) Find a maximum likelihood estimator, $\widehat{\beta}$, for β.

   (b) Find the likelihood ratio statistic for the test of $\mathrm{H}_o\colon\ \beta = \beta_o$ versus the alternative $\mathrm{H}_1\colon\ \beta \neq \beta_o$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β), for $\beta > 0$, and H_o and H_1 be as in Problem 1.

   (a) Show that the likelihood ratio statistic, $\Lambda(x_1, x_2, \ldots, x_n)$, found in part (b) of Problem 1 is of the form $e^n\ t^n e^{-nt}$, where $t = \widehat{\beta}/\beta_o$.

   (b) Let $g(t) = e^n\ t^n e^{-nt}$ for $t \geqslant 0$. Show that $g(t) \leqslant g(1) = 1$ for all $t \leqslant 0$, and sketch the graph of g.

   (c) Show that the rejection region $R\colon\ \Lambda(x_1, x_2, \ldots, x_n) \leqslant c$, for $0 < c < 1$, is equivalent to the region

$$\frac{1}{\beta_o}\overline{X}_n < c_1 \quad \text{or} \quad \frac{1}{\beta_o}\overline{X}_n > c_2,$$

   for critical values c_1 and c_2 satisfying $0 < c_1 < 1 < c_2$. Describe how you obtain c_1 and c_2 in terms of c.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β), for $\beta > 0$, and H_o and H_1 be as in Problem 1.

   Define the statistic $Y = \dfrac{2}{\beta}\displaystyle\sum_{i=1}^{n} X_i$.

   (a) Assuming that H_o is true, give the distribution of the random variable Y.

   (b) Use the information gained in part (a) to come up with values of c_1 and c_2 such that the rejection region

   $$R\colon \quad \frac{1}{\beta_o}\overline{X}_n < c_1 \quad \text{or} \quad \frac{1}{\beta_o}\overline{X}_n > c_2$$

   yields a test with significance level α.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β), for $\beta > 0$, and H_o and H_1 be as in Problem 1. Let Y denote the statistic defined in Problem 3.

   (a) If $\beta \neq \beta_o$, give the distribution of the test statistic Y.

   (b) Find an expression for the power function $\gamma(\beta)$ for the test for $\beta \neq \beta_o$.

   (c) Sketch the graph of $\gamma(\beta)$ for $\beta_o = 1$, $n = 10$ and $\alpha = 0.05$.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from an Poisson(λ), for $\lambda > 0$.

   (a) Find a maximum likelihood estimator, $\widehat{\lambda}$, for λ.

   (b) Find the likelihood ratio statistic for the test of $H_o\colon\ \lambda = \lambda_o$ versus the alternative $H_1\colon\ \lambda \neq \lambda_o$.

   (c) Show that the likelihood ratio test of H_o versus H_1 is based on the test statistic $Y = \displaystyle\sum_{i=1}^{n} X_i$. .

   (d) Obtain the distribution of Y under the assumption that H_o is true.

   (e) For $\lambda_o = 2$ and $n = 5$, find the significance level of the the test that rejects H_o if either $Y \leqslant 4$ or $Y \geqslant 7$.