Assignment #13

Due on Monday, November 16, 2009

Read Section 6.3 on *Maximum Likelihood Tests,* pp. 333–339, in Hogg, Craig and McKean.

Read Section 8.1 on *Most Powerful Tests,* pp. 419–427, in Hogg, Craig and McKean.

Read Section 8.2 on *Uniformly Most Powerful Tests,* pp. 429–435, in Hogg, Craig and McKean.

Do the following problems

1. Consider a test of the simple hypotheses

$$\mathrm{H}_o\colon \theta = \theta_o \quad \text{versus} \quad \mathrm{H}_1\colon \theta = \theta_1$$

based on a random sample from a distribution with pmf $f(x \mid \theta)$, for $x = 1, 2, \ldots, 7$. The values of the likelihood function at θ_o and θ_1 are given in the table below.

x	1	2	3	4	5	6	7
$L(\theta_o)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$L(\theta_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Use the Neyman–Pearson Lemma to find the most powerful test for H_o versus H_1 with significance level $\alpha = 0.04$. Compute the probability of Type II error for this test.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson(λ) distribution.

(a) Find the most powerful test for testing

$$\mathrm{H}_o\colon \lambda = \lambda_o \quad \text{versus} \quad \mathrm{H}_1\colon \lambda = \lambda_1,$$

for $\lambda_1 > \lambda_o$.

(b) Show that the test found in part (a) is uniformly most powerful for testing

$$\mathrm{H}_o\colon \lambda = \lambda_o \quad \text{versus} \quad \mathrm{H}_1\colon \lambda > \lambda_o.$$

3. Given a random sample, $X_1, X_2, \ldots, X_n$, from a distribution with distribution function $f(x \mid \theta)$. We say that a statistic $T = T(X_1, X_2, \ldots, X_n)$ is **sufficient** for θ is the joint distribution $f(x_1, x_2, \ldots, x_n \mid \theta)$ can be written in the form

$$f(x_1, x_2, \ldots, x_n \mid \theta) = g(T, \theta)h(x_1, x_2, \ldots, x_n),$$

for some functions $g\colon \mathbb{R}^2 \to \mathbb{R}$ and $h\colon \mathbb{R}^n \to \mathbb{R}$.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson(λ) distribution. Find a sufficient statistic for λ. Justify your answer based on the definition given above.

4. Suppose that $X_1, X_2, \ldots, X_n$ forms a random sample from distribution with distribution function $f(x \mid \theta)$.

   (a) Show that if T is a sufficient statistic for θ, then the likelihood ratio statistic for the test of
   $$\mathrm{H}_o\colon \theta = \theta_o \quad \text{versus} \quad \mathrm{H}_1\colon \theta = \theta_1$$
   is a function of T.

   (b) Explain how knowledge of the distribution of T under H_o may be used to choose a rejection region that yields the most powerful test at level α.

5. Derive a likelihood ratio test for

$$\mathrm{H}_o\colon \sigma^2 = \sigma_o^2 \quad \text{versus} \quad \mathrm{H}_1\colon \sigma^2 \neq \sigma_o^2$$

based on a sample from a normal(μ, σ^2) distribution.