Assignment #15

Due on Monday, November 30, 2009

Read Section 6.2 on *Rao-Cramér lower bound and efficiency,* pp. 319–330, in Hogg, Craig and McKean.

Background and Definitions

Crámer–Rao Information Inequality and Efficiency
Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution with distribution function $f(x \mid \theta)$, and let $W = W(X_1, X_2, \ldots, X_n)$ be an estimator for the parameter θ.

- **Information Inequality.** Put $g(\theta) = E_\theta(W)$. Then,

$$\text{var}(W) \geqslant \frac{[g'(\theta)]^2}{nI(\theta)}, \tag{1}$$

where

$$I(\theta) = \text{var}\left(\frac{\partial}{\partial\theta}\left[\ln\left(f(X \mid \theta)\right)\right]\right)$$

is the **Fisher information**. If W is unbiased, we obtain from (1) that

$$\text{var}(W) \geqslant \frac{1}{nI(\theta)}. \tag{2}$$

- **Efficient Estimator.** Let W be and unbiased estimators for θ. W is said to be **efficient** if var(W) is the lower bound in the Crámer–Rao inequality in (2); that is,

$$\text{var}(W) = \frac{1}{nI(\theta)}.$$

Do the following problems

1. Suppose that when the radius of a disc in the plane is measured, an error is made that has a normal$(0, \sigma^2)$ distribution. If n independent measurements are made, find an unbiased estimator for the area of the disc. Is this the best unbiased estimator for the area? Assume that σ^2 is known.

2. Let $X_1, X_2, \ldots, X_n$ be iid Bernoulli(p) random variables. Show that the MLE for p is an efficient estimator.

3. Let $X_1, X_2, \ldots, X_n$ be iid exponential(β) random variables, and define

$$Y = \min\{X_1, X_2, \ldots, X_n\}.$$

   Find an unbiased estimator, W, based only on Y. Compute var(W) and compare it to the variance of the sample mean, $\overline{X}_n$. Which of W or $\overline{X}_n$ is a more efficient estimator.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal(μ, σ^2) distribution. Prove that the sample mean, $\overline{X}_n$, is an efficient estimator of μ for every known $\sigma^2 > 0$.

5. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a uniform distribution over the interval $[0, \theta]$ for some parameter $\theta > 0$.

   Let $Y = \max\{X_1, X_2, \ldots, X_n\}$ and define $W = \dfrac{n+1}{n}Y$. Compute the variance W. Is W an efficient estimator of θ?