Solutions to Assignment #1

1. Let $0 < p < 1$. A random variable X is said to follow a Bernoulli(p) distribution if X takes the values 0 and 1, $p_X(0) = 1 - p$ and $p_X(1) = p$.

   Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Bernoulli(p) distribution and define the statistic $Y = X_1 + X_2 + \cdots + X_n$.

   (a) Compute the mgf of Y and use it to determine the sampling distribution of Y.

   Solution: Since the random variables $X_1, X_2, \ldots, X_n$ are independent, it follows that

$$\begin{aligned} M_Y(t) &= M_{X_1+X_2+\cdots+X_n}(t) \\ &= M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t) \\ &= \left[M_{X_1}(t)\right]^n, \end{aligned}$$

   where we have used the assumption that $X_1, X_2, \ldots, X_n$ have the same distribution. It then follows that

$$M_Y(t) = (1 - p + p\, e^t)^n,$$

   which is the mgf of a binomial(n, p) random variable. Consequently,

$$Y \sim \text{binomial}(n, p).$$

   □

   (b) Show that Y/n is an unbiased estimator of p.

   Solution: Since $Y \sim$ binomial(n, p), it follows that $E(Y) = np$. Consequently,

$$E\left(\frac{Y}{n}\right) = \frac{1}{n}E(Y) = p,$$

   which shows that Y/n is an unbiased estimator of p. □

2. A random variable, X, is said to follow an exponential distribution with parameter β, where $\beta > 0$, if X has the pdf

$$f_X(x) = \begin{cases} \dfrac{1}{\beta} e^{-x/\beta} & \text{if } x > 0; \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

We write $X \sim \text{exponential}(\beta)$.

(a) Let $\beta > 0$ and $X \sim \text{exponential}(\beta)$. Verify that the mgf of X is

$$M_X(t) = \frac{1}{1-\beta t} \quad \text{for} \quad t < \frac{1}{\beta}.$$

Solution: Compute

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, \mathrm{d}x \\ &= \int_0^{\infty} e^{tx} \frac{1}{\beta} e^{-x/\beta}\, \mathrm{d}x \\ &= \int_0^{\infty} \frac{1}{\beta} e^{-(1-\beta t)x/\beta}\, \mathrm{d}x. \end{aligned}$$

The last integral converges if and only if $1 - \beta t > 0$, or $t < \dfrac{1}{\beta}$, to

$$M_X(t) = \frac{1}{1-\beta t}.$$

□

(b) Let $\beta > 0$ and $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β) distribution. Compute the mgf of the sample mean, $\overline{X}_n$.

Solution: Compute

$$\begin{aligned} M_{\overline{X}_n}(t) &= E(e^{t\overline{X}_n}) \\ &= E\left(e^{(X_1+X_2+\cdots+X_n)\frac{t}{n}}\right) \\ &= M_{X_1+X_2+\cdots+X_n}\left(\frac{t}{n}\right) \\ &= \left[M_{X_1}\left(\frac{t}{n}\right)\right]^n, \end{aligned}$$

where we have used the assumption that $X_1, X_2, \ldots, X_n$ be a random sample. Hence, by the previous part,

$$M_{\overline{X}_n}(t) = \left[\frac{1}{1-\beta t/n}\right]^n \quad \text{for} \quad t < \frac{n}{\beta}.$$

□

(c) Let $Y_n = 2n\overline{X}_n/\beta$. Compute the mgf of Y_n.

Solution: Compute

$$
\begin{aligned}
M_{Y_n}(t) &= E(e^{tY_n}) \\
&= E\left(e^{\overline{X}_n\left(\frac{2nt}{\beta}\right)}\right) \\
&= M_{\overline{X}_n}\left(\frac{2nt}{\beta}\right) \\
&= \left[\frac{1}{1-\beta(2nt/\beta)/n}\right]^n \\
&= \left[\frac{1}{1-2t}\right]^n
\end{aligned}
$$

for $t < \dfrac{1}{2}$. □

3. Let $\Gamma\colon (0,\infty) \to \mathbb{R}$ be given by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, \mathrm{d}t \quad \text{for all} \quad x > 0. \tag{1}$$

Derive the following identities:

(a) $\Gamma(1) = 1$.

Solution: Compute

$$
\begin{aligned}
\Gamma(1) &= \int_0^\infty t^{1-1}e^{-t}\, \mathrm{d}t \\
&= \int_0^\infty e^{-t}\, \mathrm{d}t \\
&= \lim_{b\to\infty} \int_0^b e^{-t}\, \mathrm{d}t \\
&= \lim_{b\to\infty} (1 - e^{-b}) \\
&= 1.
\end{aligned}
$$

□

(b) $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.

Solution: Compute

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\,\mathrm{d}t \\ &= \lim_{b\to\infty}\int_0^b t^x e^{-t}\,\mathrm{d}t,\end{aligned}$$

where

$$\int_0^b t^x e^{-t}\,\mathrm{d}t = \left[-t^x e^{-t}\right]_0^b + \int_0^b x t^{x-1} e^{-t}\,\mathrm{d}t,$$

by virtue of integration by parts, or

$$\int_0^b t^x e^{-t}\,\mathrm{d}t = -b^x e^{-b} + x\int_0^b t^{x-1} e^{-t}\,\mathrm{d}t.$$

It then follows that

$$\lim_{b\to\infty}\int_0^b t^x e^{-t}\,\mathrm{d}t = x\int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t,$$

since

$$\lim_{b\to\infty} b^x e^{-b} = 0$$

for all $x \in \mathbb{R}$. Consequently,

$$\Gamma(x+1) = x\int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t = x\Gamma(x).$$

□

(c) $\Gamma(n+1) = n!$ for all positive integers n.

Proof: We prove the result by induction on n.
First observe that $\Gamma(1+1) = (1)\Gamma(1) = 1$ by the result of part (a). Thus, $\Gamma(1+1) = 1!$ and the result is true for $n = 1$.
Next, assume that $\Gamma(n+1) = n!$ and we prove that $\Gamma(n+2) = (n+1)!$.
Compute

$$\begin{aligned}\Gamma(n+2) &= \Gamma[(n+1)+1] \\ &= (n+1)\Gamma(n+1) \\ &= (n+1)n! \\ &= (n+1)!,\end{aligned}$$

which was to be shown. □

4. Let $\Gamma\colon (0,\infty) \to \mathbb{R}$ be as defined in (1).

   (a) Compute $\Gamma(1/2)$.
   Hint: The change of variable $t = z^2/2$ might come in handy. Recall that if $Z \sim \text{normal}(0,1)$, then its pdf is given by

$$f_Z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \quad \text{for all } z \in \mathbb{R}.$$

   Solution: Compute

$$\begin{aligned} \Gamma(1/2) &= \int_0^\infty t^{(1/2)-1} e^{-t}\, \mathrm{d}t \\ &= \int_0^\infty \frac{1}{\sqrt{t}}\, e^{-t}\, \mathrm{d}t. \end{aligned}$$

   Make the change of variable $t = z^2/2$ to get $\mathrm{d}t = z\, \mathrm{d}z$ and $\sqrt{t} = z/\sqrt{2}$. It then follows that

$$\begin{aligned} \Gamma(1/2) &= \int_0^\infty \frac{\sqrt{2}}{z}\, e^{-z^2/2} z\, \mathrm{d}z \\ &= 2\sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, \mathrm{d}z \\ &= \sqrt{\pi} \int_{-\infty}^\infty f_Z(z)\, \mathrm{d}z \\ &= \sqrt{\pi}, \end{aligned}$$

   which was to be shown. □

   (b) Compute $\Gamma(3/2)$.

   Solution: Use the result from part (b) of Problem 3 to get that

$$\Gamma(3/2) = \Gamma[(1/2)+1] = (1/2)\Gamma(1/2) = \sqrt{\pi}/2.$$

□

5. Use the results of Problems 3 and 4 to derive the identity:

$$\Gamma\left(\frac{k}{2}\right) = \frac{\Gamma(k)\sqrt{\pi}}{2^{k-1}\ \Gamma\left(\frac{k+1}{2}\right)} \tag{2}$$

for every positive, odd integer k.

Proof. We proceed by induction on odd k.

For $k = 1$ we have, by part (a) of Problem 4 that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{\Gamma(1)\sqrt{\pi}}{2^{1-1}\ \Gamma\left(\frac{1+1}{2}\right)}$$

since $\Gamma(1) = 1$ by part (a) of Problem 3. Thus, the result is true for $k = 1$.

Next assume that (2) for an odd integer k; we show that the result is true for the next odd integer $k + 2$.

Using part (b) of Problem 3 we get

$$\begin{aligned} \Gamma\left(\frac{k+2}{2}\right) &= \Gamma\left(\frac{k}{2}+1\right) \\ &= \frac{k}{2}\cdot\Gamma\left(\frac{k}{2}\right); \end{aligned}$$

so that, by the inductive hypothesis (2),

$$\begin{aligned} \Gamma\left(\frac{k+2}{2}\right) &= \frac{k}{2}\cdot\frac{\Gamma(k)\sqrt{\pi}}{2^{k-1}\ \Gamma\left(\frac{k+1}{2}\right)} \\ &= \frac{\Gamma(k+1)\sqrt{\pi}}{2^{k}\ \Gamma\left(\frac{k+1}{2}\right)} \\ &= \frac{(k+1)\Gamma(k+1)\sqrt{\pi}}{2^{k}\ (k+1)\Gamma\left(\frac{k+1}{2}\right)} \\ &= \frac{\Gamma(k+2)\sqrt{\pi}}{2^{k+1}\ \frac{k+1}{2}\Gamma\left(\frac{k+1}{2}\right)} \\ &= \frac{\Gamma(k+2)\sqrt{\pi}}{2^{k+1}\ \Gamma\left(\frac{k+1}{2}+1\right)} \\ &= \frac{\Gamma(k+2)\sqrt{\pi}}{2^{k+1}\ \Gamma\left(\frac{k+3}{2}\right)}. \end{aligned}$$

Hence

$$\Gamma\left(\frac{k+2}{2}\right) = \frac{\Gamma(k+2)\sqrt{\pi}}{2^{(k+2)-1}\ \Gamma\left(\frac{(k+2)+1}{2}\right)},$$

which shows that the result is true for $k+2$. $\square$