Solutions to Assignment #2

1. The reason that the function $M_X(t)$ is called the moment generating function for random variable X is that the n^{th} derivative of $M_X(t)$ at $t = 0$ is $E(X^n)$, the n^{th} moment of the random variable X; that is,

$$M_X^{(n)}(0) = E(X^n) \quad \text{for} \ \ n = 1, 2, 3, \ldots \tag{1}$$

(a) Verify (1) for the case in which X is continuous with pdf f_X. What assumptions do you need to make about the mgf in your derivation?

Solution: For the case of a continuous random variable, X, with pdf f_X, its mgf is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, \mathrm{d}x,$$

for values of t in some interval around 0. Assuming that

$$\int_{-\infty}^{\infty} |x|^n e^{tx} f_X(x)\, \mathrm{d}x < \infty$$

for all t in some interval around 0, since the functions

$$(x, t) \mapsto x^n e^{tx}$$

are continuous, it follows by differentiating under the integral sign with respect to t that

$$M_X^{(n)}(t) = \int_{-\infty}^{\infty} x^n\ e^{tx} f_X(x)\, \mathrm{d}x \quad \text{for all } n = 1, 2, 3, \ldots$$

and for t in an interval around 0. Consequently,

$$M_X^{(n)}(0) = \int_{-\infty}^{\infty} x^n\ f_X(x)\, \mathrm{d}x = E(X^n) \quad \text{for all } n = 1, 2, 3, \ldots$$

which was to be shown. □

(b) Show that if the mgf of X exists on some interval around 0, then

$$\mathrm{var}(X) = M_X''(0) - \left[M_X'(0)\right]^2$$

Solution: For this problem, we also need to assume that the mgf of X is twice differentiable at 0. Then,

$$\begin{aligned} \text{var}(X) &= E[(X-\mu_X)^2] \\ &= E(X^2 - 2\mu_X X + \mu_X^2), \end{aligned}$$

where $\mu_X = E(X)$. Thus, by the linearity of the expectation operator,

$$\begin{aligned} \text{var}(X) &= E(X^2) - 2\mu_X E(X) + \mu_X^2 E(1) \\ &= E(X^2) - 2\mu_X E(X) + \mu_X^2 E(1) \\ &= E(X^2) - 2\mu_X \mu_X + \mu_X^2 \\ &= E(X^2) - \mu_X^2 \\ &= E(X^2) - [E(X)]^2 \\ &= M_X''(0) - [M_X'(0)]^2, \end{aligned}$$

which was to be shown. □

2. Let $\lambda > 0$. A random variable X is said to follow a Poisson(λ) distribution if X takes the values $0, 1, 2, 3, \ldots$ and the pmf of X is given by

$$p_X(k) = \frac{\lambda^k}{k!}\, e^{-\lambda} \quad \text{for all} \quad k = 0, 1, 2, 3, \ldots$$

Compute the mgf of a Poisson(λ) random variable, X. For which values of t is the mgf defined?

Solution: Compute

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{k=0}^{\infty} e^{tk} p_X(k) \\ &= \sum_{k=0}^{\infty} (e^t)^k \frac{\lambda^k}{k!}\, e^{-\lambda}. \end{aligned}$$

It then follow that the mgf of $X \sim \text{Poisson}(\lambda)$ is

$$\begin{aligned} M_X(t) &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \\ &= e^{-\lambda}\, e^{\lambda e^t} \\ &= e^{\lambda(e^t-1)}, \end{aligned}$$

for all $t \in \mathbb{R}$. $\square$

3. Use the result of Problem 2 to compute the mean and variance of a Poisson(λ) distribution. What do you discover?

***Solution**:* Differentiating the mgf of X obtained in Problem 2 with respect to t, we get

$$M_X'(t) = \lambda e^t\, e^{\lambda(e^t-1)},$$

and

$$M_X''(t) = \lambda e^t\, e^{\lambda(e^t-1)} + \lambda^2 e^{2t}\, e^{\lambda(e^t-1)}.$$

We then get that the expected value of X is

$$E(X) = M_X'(0) = \lambda,$$

and the second moment of X is

$$E(X^2) = M_X''(0) = \lambda + \lambda^2.$$

Consequently, by the result in part (b) of Problem 1, the variance of X is

$$\text{var}(X) = E(X^2) - \lambda^2 = \lambda.$$

Thus, the expected value and variance of a Poisson random variable are the same. $\square$

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson(λ) distribution. Define $Y_n = X_1 + X_2 + \cdots + X_n$. Give the sampling distribution for Y_n. What do you discover?

Solution: Compute the mgf of Y_n, $M_{Y_n}(t) = E(e^{tY_n})$, to get that

$$M_{Y_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t),$$

where we have used the independence assumption. Thus, since the random variables $X_1, X_2, \ldots, X_n$ are identically distributed, it follows from the result of Problem 2 that

$$M_{Y_n}(t) = \left(e^{\lambda(e^t-1)}\right)^n = e^{n\lambda(e^t-1)},$$

which is the mgf of a Poisson($n\lambda$) random variable. It follows that Y_n has a Poisson distribution with parameter $n\lambda$. □

5. Let $X_1, X_2, X_3 \ldots$ be a sequence of random variable satisfying $X_n \sim$ binomial(n, p) for all n. Assume also that $np = \lambda$, where λ is a fixed parameter.

   Compute $M_{X_n}(t)$ for all n and determine the limit

$$\lim_{n\to\infty} M_{X_n}(t).$$

   What do you discover?

   Hint: Observe that $p = \dfrac{\lambda}{n} \to 0$ as $n \to \infty$ since λ is assumed to be fixed.

Solution: Compute

$$M_{X_n}(t) = (1 - p + p\, e^t)^n = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n}\, e^t\right)^n,$$

or

$$M_{X_n}(t) = \left(1 + \frac{\lambda(e^t-1)}{n}\right)^n.$$

It then follows that

$$\lim_{n\to\infty} M_{X_n}(t) = \lim_{n\to\infty}\left(1 + \frac{\lambda(e^t-1)}{n}\right)^n = e^{\lambda(e^t-1)},$$

which is the mgf of a Poisson(λ) random variable. Note that we have used the definition of e^u as

$$e^u = \lim_{n\to\infty}\left(1 + \frac{u}{n}\right)^n \quad \text{for all} \quad u \in \mathbb{R}.$$

□