Assignment #3

Due on Wednesday, September 16, 2009

Read Section 2.2 on *Transformations: Bivariate Random Variables,* pp. 84–92, in Hogg, Craig and McKean.

Do the following problems

1. Let X and Y be independent continuous random variables with pdfs f_X and f_Y, respectively. Let $W = X + Y$ and show that the pdf for W is given by

$$f_W(w) = \int_{-\infty}^{+\infty} f_X(u) f_Y(w-u)\ \mathrm{d}u \tag{1}$$

for all $w \in \mathbb{R}$. This is known as the *convolution* of f_X and f_Y.

Suggestion: To evaluate the double integral defining $P(X + Y \leq z)$, make the change of variables $u = x$ and $v = x + y$. Observe that with this change of variables, the region of integration in the uv–plane becomes:

$$\{(u, v) \in \mathbb{R}^2 \mid -\infty < u < \infty, -\infty < v < z\}.$$

Refer to pages 86 and 87 in the text on how to perform a change of variables for a double integral.

2. Let $X \sim$ exponential(2) and $Y \sim \chi^2(1)$ be independent random variables. Define $W = X + Y$. Use the convolution formula in (1) to find the pdf of W.

3. We use the notation $f_X * f_Y$ to denote the convolution of the two pdfs f_X and f_Y as defined in (1); that is,

$$f_X * f_Y(w) = \int_{-\infty}^{+\infty} f_X(u) f_Y(w-u)\ \mathrm{d}u \quad \text{for all } w \in \mathbb{R}.$$

Verify that convolution is a symmetric operation; that is,

$$f_X * f_Y = f_Y * f_X.$$

4. Suppose that the pdf of a random variable, W, is the convolution of two pdfs f_X and f_Y for two random variables, X and Y.

   Verify that
   $$M_W(t) = M_X(t) \cdot M_Y(t)$$
   for t in some interval around 0 where the mgfs of X and Y are both defined; that is, the moment generating function of a convolution is the product of the moment generating functions.

5. Let α and β denote positive real numbers and define $f(x) = Cx^{\alpha-1}e^{-x/\beta}$ for $x > 0$ and $f(x) = 0$ for $x \leqslant 0$, where C denotes a positive real number.

   (a) Find the value of C so that f is the pdf for some distribution.

   (b) For the value of C found in part (a), let f denote the pdf of a random variable X. Compute the mgf of X.

   Hint: The pdf found in part (a) is related to the Gamma function.