Solutions to Assignment #8

1. Let the random vector (X_1, X_2) have a multinomial distribution with parameters n, p_1, p_2.

   (a) Give the marginal distributions for X_1 and X_2 and compute $E(X_i)$ for $i = 1, 2$.

   ***Solution**:* Compute, for $n_1 = 0, 1, 2, \ldots, n$,

$$\begin{aligned} p_{X_1}(n_1) &= \sum_{\substack{n_2 \\ n_2 = n - n_1}} \frac{n!}{n_1! n_2!} p_1^{n_1} p_2^{n_2} \\ &= \frac{n!}{n_1!(n-n_1)!} p_1^{n_1} (1-p_1)^{n-n_1}, \end{aligned}$$

   since $p_1 + p_2 = 1$. This is the pmf for a binomial(n, p_1) random variable. Hence, $X_1 \sim$ binomial(n, p_1). Similarly, $X_2 \sim$ binomial(n, p_2). It then follows that

$$E(X_i) = np_i \quad \text{for } i = 1, 2.$$

   □

   (b) Show that X_1 and X_2 are not independent and compute the covariance, cov(X_1, X_2), of X_1 and X_2.

   ***Solution**:* Note that

$$\begin{aligned} \mathrm{P}(X_1 = n_1, X_2 = n_2) &= \mathrm{P}(X_1 = n_1, X_1 = n - n_2) \\ &= \begin{cases} \dfrac{n!}{n_1! n_2!} p_1^{n_1} p_2^{n_2} & \text{if } n_1 + n_2 = n \\ \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

   while

$$p_{X_1}(n_1) p_{(X_2}(n_2) = \frac{n!}{n_1! n_2!} p_1^{n_1} \frac{n!}{n_1! n_2!} p_2^{n_1}$$

   for $n_1 + n_2 = n$. Thus, X_1 and X_1 are not independent.

To find $\text{cov}(X_1, X_2)$ compute

$$\begin{aligned}
\text{cov}(X_1, X_2) &= E[(X_1 - np_1)(X_2 - np_2)] \\
&= E[X_1(X_2 - np_2) - np_1(X_2 - np_2)] \\
&= E[X_1X_2 - np_2X_1] - np_1E(X_2 - np_2) \\
&= E[X_1X_2] - np_2E[X_1] \\
&= E[X_1X_2] - n^2p_2p_1 \\
&= E[X_1X_2] - n^2(1 - p_1)p_1 \\
&= E[X_1X_2] - n^2p_1 + n^2p_1^2 \\
&= E[X_1X_2] - n^2p_1 + (E(X_1))^2,
\end{aligned}$$

where

$$\begin{aligned}
E[X_1X_2] &= E[X_1(n - X_1)] \\
&= E[nX_1 - X_1^2] \\
&= nE[X_1] - E[X_1^2] \\
&= n^2p_1 - E[X_1^2].
\end{aligned}$$

It then follows that

$$\begin{aligned}
\text{cov}(X_1, X_2) &= -E(X_1^2) + (E(X_1))^2 \\
&= -\text{var}(X_1) \\
&= -np_1(1 - p_1) \\
&= -np_1p_2.
\end{aligned}$$

□

2. Given two random variables, X and Y, the joint moment generating function of X and Y, denoted by $M_{(X,Y)}(t_1, t_2)$, is defined to be be

$$M_{(X,Y)}(t_1, t_2) = E(e^{t_1X+t_2Y})$$

for (t_1, t_2) in some neighborhood of the origin in $\mathbb{R}^2$.

Let the random vector (X_1, X_2) have a multinomial distribution with parameters n, p_1, p_2.

(a) Compute the joint mgf of (X_1, X_2).

Solution: Compute

$$\begin{aligned} M_{(X_1,X_2)}(t_1,t_2) &= E(e^{t_1X_1+t_2X_2}) \\ &= \sum_{\substack{n_1,n_2 \\ n_1+n_2=n}} e^{t_1n_1+t_2n_2}\frac{n!}{n_1!n_2!}p_1^{n_1}p_2^{n_2} \\ &= \sum_{\substack{n_1,n_2 \\ n_1+n_2=n}} \frac{n!}{n_1!n_2!}(p_1e^{t_1})^{n_1}(p_2e^{t_2})^{n_2} \\ &= (p_1e^{t_1}+p_2e^{t_2})^n, \end{aligned}$$

by the binomial theorem. □

(b) Verify that $\operatorname{cov}(X_1, X_2) = \dfrac{\partial^2 M}{\partial t_1 \partial t_2}(0,0) - \dfrac{\partial M}{\partial t_1}(0,0)\dfrac{\partial M}{\partial t_2}(0,0)$, where $M = M_{(X_1,X_2)}$.

Solution: Write $M(t_1,t_2) = (p_1e^{t_1}+p_2e^{t_2})^n$ and compute

$$\begin{aligned} \frac{\partial M}{\partial t_1}(t_1,t_2) &= n(p_1e^{t_1}+p_2e^{t_2})^{n-1}\cdot p_1e^{t_1} \\ &= np_1e^{t_1}(p_1e^{t_1}+p_2e^{t_2})^{n-1}. \end{aligned}$$

Similarly,

$$\frac{\partial M}{\partial t_2}(t_1,t_2) = np_2e^{t_2}(p_1e^{t_1}+p_2e^{t_2})^{n-1}.$$

Differentiating one more time we get

$$\begin{aligned} \frac{\partial^2 M}{\partial t_1 \partial t_2}(t_1,t_2) &= np_2e^{t_2}\cdot(n-1)p_1e^{t_1}(p_1e^{t_1}+p_2e^{t_2})^{n-2} \\ &= n(n-1)p_1p_2e^{t_1+t_2}(p_1e^{t_1}+p_2e^{t_2})^{n-2}. \end{aligned}$$

We then have that

$$\begin{aligned}\frac{\partial^2 M}{\partial t_1 \partial t_2}(0,0) &= n(n-1)p_1p_2(p_1+p_2)^{n-2} \\ &= n(n-1)p_1p_2,\end{aligned}$$

since $p_1 + p_2 = 1$.
Similarly

$$\begin{aligned}\frac{\partial M}{\partial t_1}(0,0)\frac{\partial M}{\partial t_2}(0,0) &= n^2p_1p_2(p_1+p_2)^{2(n-1)} \\ &= n^2p_1p_2.\end{aligned}$$

We then have that

$$\frac{\partial^2 M}{\partial t_1 \partial t_2}(0,0) - \frac{\partial M}{\partial t_1}(0,0)\frac{\partial M}{\partial t_2}(0,0) = -np_1p_2,$$

which is the value for $\text{cov}(X_1, X_2)$ that we got in part (b) of Problem 1. □

3. Let X_1 and X_2 be independent Poisson(λ) random variables. For a fixed value of n ($n = 0, 1, 2, 3, \ldots$), determine the conditional distribution of X_1 given that $X_1 + X_2 = n$.

Solution: Let $Y = X_1 + X_2$. Then, since X_1 and X_2 are independent Poisson(λ) random variables, $Y \sim$ Poisson(2λ); so that the pmf of Y is

$$p_{_Y}(m) = \frac{(2\lambda)^m}{m!}e^{-2\lambda} \quad \text{for} \quad m = 0, 1, 2, \ldots$$

We want to determine the conditional distribution of X_1 given $Y = n$. For $k = 0, 1, 2, \ldots, n$, compute

$$\begin{aligned}p_{_{X_1|Y}}(k \mid n) &= \frac{\text{P}(X_1 = k, Y = n)}{\text{P}(Y = n)} \\ &= \frac{\text{P}(X_1 = k, X_1 + X_2 = n)}{p_{_Y}(n)} \\ &= \frac{\text{P}(X_1 = k, X_2 = n-k)}{p_{_Y}(n)} \\ &= \frac{p_{_{X_1}}(k) \cdot p_{_{X_2}}(n-k)}{p_{_Y}(n)},\end{aligned}$$

by the independence of X_1 and X_2, where

$$p_{X_1}(k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

and

$$p_{X_2}(n-k) = \frac{\lambda^{n-k}}{(n-k)!}e^{-\lambda}.$$

We then have that

$$\begin{aligned} p_{X_1|Y}(k \mid n) &= \frac{p_{X_1}(k)\cdot p_{X_2}(n-k)}{p_Y(n)} \\ &= \frac{\dfrac{\lambda^k}{k!}e^{-\lambda}\cdot\dfrac{\lambda^{n-k}}{(n-k)!}e^{-\lambda}}{\dfrac{(2\lambda)^n}{n!}e^{-2\lambda}} \\ &= \frac{n!}{k!(n-k)!}\left(\frac{1}{2}\right)^n, \end{aligned}$$

which is the pmf of a binomial$(n, 1/2)$ random variable. It then follows that

$$(X_1 \mid Y = n) \sim \text{binomial}(n, 1/2).$$

□

4. Let $X_1, X_2, \ldots, X_k$ be independent random variables satisfying $X_i \sim \text{Poisson}(\lambda_i)$ for positive parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$. For a fixed value of n ($n = 0, 1, 2, 3, \ldots$), determine the conditional distribution of the random vector $(X_1, X_2, \ldots, X_k)$ given that $X_1 + X_2 + \cdots + X_k = n$.

    ***Solution**:* Write $Y = X_1 + X_2 + \cdots + X_k$; then, since $X_1, X_2, \ldots, X_k$ are independent Poisson(λ_i) random variables, respectively, $Y \sim \text{Poisson}(\lambda)$, where

$$\lambda = \sum_{j=1}^{k} \lambda_j.$$

We want to determine the conditional distribution of

$$(X_1, X_2, \ldots, X_k) \mid Y = n.$$

For nonnegative integers $n_1, n_2, \ldots, n_k$ such that $n_1+n_2+\cdots+n_k = n$, compute

$$
\begin{aligned}
p_{(X_1,X_2,\ldots,X_k)|Y}&(n_1,n_2,\ldots,n_k) \mid n) \\
&= \frac{\mathrm{P}(X_1 = n_1, X_2 = n_2, \ldots, X_k = n_k, Y = n)}{\mathrm{P}(Y = n)} \\
&= \frac{\mathrm{P}(X_1 = n_1, X_2 = n_2, \ldots, X_k = n - n_1 - n_2 - \cdots - n_{k-1})}{p_Y(n)} \\
&= \frac{p_{X_1}(n_1)\cdot p_{X_2}(n_2)\cdots p_{X_k}(n - n_1 - n_2 - \cdots - n_{k-1})}{p_Y(n)},
\end{aligned}
$$

by the independence of $X_1, X_2, \ldots, X_k$. We then have that

$$
\begin{aligned}
p_{(X_1,X_2,\ldots,X_k)|Y}&(n_1,n_2,\ldots,n_k) \mid n) \\
&= \frac{\dfrac{\lambda_1^{n_1}}{n_1!}e^{-\lambda_1}\cdot\dfrac{\lambda_2^{n_2}}{n_2!}e^{-\lambda_2}\cdots\dfrac{\lambda_k^{n-n_1-n_2-\cdots-n_{k-1}}}{(n-n_1-n_2-\cdots-n_{k-1})!}e^{-\lambda_k}}{\dfrac{\lambda^n}{n!}e^{-\lambda}} \\
&= \frac{n!}{n_1!n_2!\cdots n_k!}\cdot\frac{\lambda_1^{n_1}\lambda_2^{n_2}\cdots\lambda_k^{n_k}}{\lambda^n} \\
&= \frac{n!}{n_1!n_2!\cdots n_k!}\cdot\frac{\lambda_1^{n_1}\lambda_2^{n_2}\cdots\lambda_k^{n_k}}{\lambda^{n_1+n_2+\cdots+n_k}} \\
&= \frac{n!}{n_1!n_2!\cdots n_k!}\cdot\left(\frac{\lambda_1}{\lambda}\right)^{n_1}\cdot\left(\frac{\lambda_2}{\lambda}\right)^{n_2}\cdots\left(\frac{\lambda_k}{\lambda}\right)^{n_k},
\end{aligned}
$$

which is the pmf of a multinomial $\left(n, \dfrac{\lambda_1}{\lambda}, \dfrac{\lambda_2}{\lambda}, \ldots, \dfrac{\lambda_k}{\lambda}\right)$ random vector. Hence, $(X_1, X_2, \ldots, X_k) \mid Y = n$ has a multinomial $\left(n, \dfrac{\lambda_1}{\lambda}, \dfrac{\lambda_2}{\lambda}, \ldots, \dfrac{\lambda_k}{\lambda}\right)$ distribution, where $\lambda = \displaystyle\sum_{j=1}^{k}\lambda_j$. □

5. Let the random vector (X_1, X_2) have a multinomial distribution with parameters n, p_1, p_2. Define the random variable $Q = \dfrac{(X_1 - np_1)^2}{np_1} + \dfrac{(X_2 - np_2)^2}{np_2}$. Show that for large values of n, Q has, approximately, a $\chi^2(1)$ distribution.

   Suggestion Use the result of part (b) in Problem 1 and apply the Central Limit Theorem.

***Solution**:* Since X_1binomial(n, p_1), the random variable

$$\frac{X_1 - np_1}{\sqrt{np_1(1-p_1)}}$$

has an approximate normal$(0,1)$ distribution for large values of n. Consequently, for large values of n,

$$\frac{(X_1 - np_1)^2}{np_1(1-p_1)}$$

has an approximate $\chi^2(1)$ distribution.

Note that we can write

$$\begin{aligned}
\frac{(X_1 - np_1)^2}{np_1(1-p_1)} &= \frac{(X_1 - np_1)^2(1-p_1) + (X_1 - np_1)^2 p_1}{np_1(1-p_1)} \\
&= \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_1 - np_1)^2}{n(1-p_1)} \\
&= \frac{(X_1 - np_1)^2}{np_1} + \frac{(n - X_2 - np_1)^2}{n(1-p_1)} \\
&= \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_2 - n(1-p_1))^2}{n(1-p_1)} \\
&= \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_2 - np_2)^2}{np_2},
\end{aligned}$$

which is the Pearson Chi–Square statistic, Q, for $k = 2$. We have therefore proved that, for large values of n, the random variable

$$Q = \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_2 - np_2)^2}{np_2}$$

has an approximate $\chi^2(1)$ distribution. □