Solutions to Review Problems for Exam #1

1. Let X and Y be independent normal$(0,1)$ random variables and define

$$W = \frac{(X-Y)^2}{2}.$$

Give the distribution of W.

Suggestion: First, determine the distribution of $X - Y$.

***Solution**:* Since X and Y are independent, it follows that

$$\begin{aligned} M_{X-Y}(t) &= M_X(t)\cdot M_Y(-t) \\ &= e^{t^2/2}\cdot e^{(-t)^2/2} \\ &= e^{2t^2/2}, \end{aligned}$$

which is the mgf of a normal$(0,2)$ distribution. Thus, $X-Y$ has a normal distribution with mean 0 and variance 2. It then follows that

$$\frac{X-Y}{\sqrt{2}} \sim \text{normal}(0,1),$$

and therefore

$$\frac{(X-Y)^2}{2} \sim \chi^2(1).$$

Hence, W has a χ^2 distribution with one degree of freedom. □

2. Let X denote a random variable with mgf $M_X(t)$ defined on some interval around 0. Put $S(t) = \ln(M_X(t))$ and prove that

$$S'(0) = E(X) \quad \text{and} \quad S''(0) = \text{var}(X).$$

***Solution**:* Differentiating with respect to t we obtain

$$S'(t) = \frac{1}{M_X(t)} M_X'(t),$$

from which we get that

$$S'(0) = \frac{1}{M_X(0)} M_X'(0) = E(X).$$

Differentiating one more time with respect to t we obtain

$$S''(t) = \frac{M_X(t)M_X''(t) - M_X'(t)M_X'(t)}{[M_X(t)]^2}.$$

Consequently,

$$S''(0) = M_X(0)M_X''(0) - [M_X'(0)]^2 = E(X^2) - [E(X)]^2 = \text{var}(X).$$

□

3. A median of a distribution of a random variable, X, is a value, m, such that

$$\text{P}(X \leqslant m) \geqslant \frac{1}{2} \quad \text{and} \quad \text{P}(X \geqslant m) \geqslant \frac{1}{2}.$$

(a) Prove that if X is continuous with pdf f_X, then a median m satisfies

$$\int_{-\infty}^{m} f_X(x)\ \mathrm{d}x = \int_{m}^{+\infty} f_X(x)\ \mathrm{d}x = \frac{1}{2}.$$

Solution: Note the

$$\text{P}(X \leqslant m) = \int_{-\infty}^{m} f_X(x)\ \mathrm{d}x \quad \text{and} \quad \text{P}(X \geqslant m) = \int_{m}^{\infty} f_X(x)\ \mathrm{d}x,$$

from which we get that

$$\int_{-\infty}^{m} f_X(x)\ \mathrm{d}x \geqslant \frac{1}{2} \quad \text{and} \quad \int_{m}^{\infty} f_X(x)\ \mathrm{d}x \geqslant \frac{1}{2}.$$

Also,

$$\int_{-\infty}^{m} f_X(x)\ \mathrm{d}x + \int_{m}^{\infty} f_X(x)\ \mathrm{d}x = 1,$$

from which we get that

$$\int_{-\infty}^{m} f_X(x)\ \mathrm{d}x = 1 - \int_{m}^{\infty} f_X(x)\ \mathrm{d}x \leqslant 1 - \frac{1}{2} = \frac{1}{2}.$$

Thus,

$$\int_{-\infty}^{m} f_X(x)\ \mathrm{d}x \geqslant \frac{1}{2} \quad \text{and} \quad \int_{-\infty}^{m} f_X(x)\ \mathrm{d}x \leqslant \frac{1}{2},$$

which imply that

$$\int_{-\infty}^{m} f_X(x)\, \mathrm{d}x = \frac{1}{2}.$$

Similarly,

$$\int_{m}^{\infty} f_X(x)\, \mathrm{d}x = \frac{1}{2}.$$

□

(b) Let $\beta > 0$ and $X \sim \text{exponential}(\beta)$. Compute a median of X. Is the value you obtained the only median of the distribution? How does your answer compare with the mean of the distribution?

Solution: The pdf of X is

$$f_X(x) = \begin{cases} \dfrac{1}{\beta}\, e^{-x/\beta} & \text{if } x > 0; \\ \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

To find a median, m, for the distribution of X, we need to solve

$$\int_{-\infty}^{m} f_X(x)\, \mathrm{d}x = \frac{1}{2},$$

by the result of part (a). We then see that $m > 0$ and

$$\int_{0}^{m} \frac{1}{\beta}\, e^{-x/\beta}\, \mathrm{d}x = \frac{1}{2},$$

from which we get that

$$1 - e^{-m/\beta} = \frac{1}{2}.$$

Solving for m we see that

$$m = (\ln 2)\beta.$$

Thus the median of the distribution is smaller than the expected value, or mean, of the distribution in this case. □

(c) Show that if X is a continuous random variable, and m is a median of the the distribution of X, then m a number which minimizes the expression

$$h(t) = E(|X - t|) \quad \text{for } t \in \mathbb{R}.$$

That is, $E(|X - m|) = \min_{t \in \mathbb{R}} E(|X - t|)$.

Solution: Write

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} |x-t| f_X(x)\ \mathrm{d}x \\ &= \int_{-\infty}^{t} |x-t| f_X(x)\ \mathrm{d}x + \int_{t}^{\infty} |x-t| f_X(x)\ \mathrm{d}x \\ &= \int_{-\infty}^{t} -(x-t) f_X(x)\ \mathrm{d}x + \int_{t}^{\infty} (x-t) f_X(x)\ \mathrm{d}x \\ &= t\left(\int_{-\infty}^{t} f_X(x)\ \mathrm{d}x - \int_{t}^{\infty} f_X(x)\ \mathrm{d}x\right) \\ &\qquad + \int_{t}^{\infty} x f_X(x)\ \mathrm{d}x - \int_{-\infty}^{t} x f_X(x)\ \mathrm{d}x. \end{aligned}$$

Taking the derivative with respect to t we obtain

$$\begin{aligned} h'(t) &= \int_{-\infty}^{t} f_X(x)\ \mathrm{d}x - \int_{t}^{\infty} f_X(x)\ \mathrm{d}x + 2t f_X(t) \\ &\qquad -t f_X(t) - t f_X(t) \\ &= \int_{-\infty}^{t} f_X(x)\ \mathrm{d}x - \int_{t}^{\infty} f_X(x)\ \mathrm{d}x, \end{aligned}$$

where we have used the product rule and the Fundamental Theorem of Calculus. Similarly,

$$h''(t) = 2 f_X(t) \geqslant 0, \quad \text{for all } t \in \mathbb{R}.$$

It then follows that a critical point of h is a minimizer of h and satisfies

$$\int_{-\infty}^{t} f_X(x)\ \mathrm{d}x - \int_{t}^{\infty} f_X(x)\ \mathrm{d}x = 0,$$

or

$$\int_{-\infty}^{t} f_X(x)\ \mathrm{d}x = \int_{t}^{\infty} f_X(x)\ \mathrm{d}x,$$

which is the definition of a median for the distribution of X. Hence, $E(|X-t|)$ is minimized when $t = m$. □

4. Give a random variable, X, of expected value μ and variance σ^2, the *skewness* of the distribution of X, denoted Skew(X), is defined to be

$$\text{Skew}(X) = \frac{E(X-\mu)^3}{\sigma^3}.$$

Observe that

$$\begin{aligned} E(X-\mu)^3 &= E[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3] \\ &= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3 \\ &= E(X^3) - 3\mu E(X^2) + 2\mu^3. \end{aligned}$$

Thus, using $\text{var}(X) = E(X^2) - [E(X)]^2$, we get that $E(X^2) = \sigma^2 + \mu^2$ so that

$$E(X-\mu)^3 = E(X^3) - 3\mu(\sigma^2+\mu^2) + 2\mu^3,$$

or

$$E(X-\mu)^3 = E(X^3) - 3\mu\sigma^2 - \mu^3. \tag{1}$$

We will use this equation to evaluate skewness in parts (a) and (b).

(a) Let $\beta > 0$ and $X \sim \text{exponential}(\beta)$. Compute the skewness of X.

***Solution**:* According to the formula in (1) we need to compute the third moment of $X \sim \text{exponential}(\beta)$. We can do this by looking up the mgf of X:

$$M_X(t) = \frac{1}{1-\beta t} \quad \text{for} \quad t < \frac{1}{\beta}.$$

Differentiating with respect to t we have that

$$M_X'(t) = \frac{\beta}{(1-\beta t)^2},$$

$$M_X''(t) = \frac{2\beta^2}{(1-\beta t)^3},$$

and

$$M_X'''(t) = \frac{6\beta^3}{(1-\beta t)^4},$$

for $t < \dfrac{1}{\beta}$. We then have that the third moment of X is

$$E(X^3) = M_X'''(0) = 6\beta^3.$$

Consequently, using (1) we have that

$$E(X-\mu)^3 = 2\beta^3$$

since $\mu = \beta$ and $\sigma^2 = \beta^2$. We therefore have that the skewness of $X \sim \text{exponential}(\beta)$ is

$$\text{Skew}(X) = \frac{E(X-\mu)^3}{\sigma^3} = \frac{2\beta^3}{\beta^3} = 2.$$

□

(b) Let $Z \sim \text{normal}(0,1)$. Compute the skewness of Z.

***Solution**:* The moment generating function of Z is $M_Z(t) = e^{t^2/2}$ and, therefore, the moments of Z are

$$E(Z) = 0,$$

$$E(Z^2) = 1,$$

and

$$E(Z^3) = 0$$

since

$$M_Z'''(t) = t(3+t^2)e^{t^2/2} \quad \text{for all } t \in \mathbb{R}.$$

it then follows that $E(Z-\mu)^3 = 0$, since $\mu = 0$ in this case. Consequently, the skewness of $Z \sim \text{normal}(0,1)$ is

$$\text{Skew}(Z) = \frac{E(Z-\mu)^3}{\sigma^3} = 0.$$

□

5. Let X and Y be independent, normal$(0,\sigma^2)$ random variables, and define

$$U = X^2 + Y^2 \quad \text{and} \quad V = \frac{X}{\sqrt{U}}.$$

(a) Find the joint pdf, $f_{(U,V)}$, of U and V.

Solution: First we compute the joint cdf of U and V,

$$F_{(U,V)}(u,v) = \mathrm{P}(U \leqslant u, V \leqslant v) \quad \text{for} \quad u > 0 \text{ and } -1 < v < 1,$$

or

$$\begin{aligned} F_{(U,V)}(u,v) &= \mathrm{P}(X^2+Y^2 \leqslant u, X/\sqrt{X^2+Y^2} \leqslant v) \\ &= \iint_{R_{u,v}} f_{(X,Y)}(x,y)\ \mathrm{d}x\ \mathrm{d}y, \end{aligned}$$

where the joint pdf of X and Y is

$$f_{(X,Y)}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^+y^2)/2\sigma^2} \quad \text{for} \quad (x,y) \in \mathbb{R}^2,$$

since X and Y are independent normal$(0,\sigma^2)$ random variables, and $R_{u,v}$ is the region in the xy–plane defined by

$$x^2+y^2 \leqslant u \quad \text{and} \quad x \leqslant v\sqrt{x^2+y^2}$$

for $u > 0$ and $-1 < v < 1$.
Next, make the change of variables

$$r = \sqrt{x^2+y^2} \quad \text{and} \quad w = \frac{x}{\sqrt{x^2+y^2}}.$$

We then have that

$$x = rw \quad \text{and} \quad x^2+y^2 = r^2.$$

Solving for y in the previous two equations, we see that we have two possibilities

$$y = r\sqrt{1-w^2} \quad \text{or} \quad y = -r\sqrt{1-w^2}.$$

Thus, the region $R_{u,v}$ is divided into two disjoint regions $R^+_{u,v}$ and $R^-_{u,v}$ corresponding to $y > 0$ and $y < 0$, respectively. We then have that

$$F_{(U,V)}(u,v) = \iint_{R^+_{u,v}} f_{(X,Y)}(x,y)\ \mathrm{d}x\ \mathrm{d}y + \iint_{R^-_{u,v}} f_{(X,Y)}(x,y)\ \mathrm{d}x\ \mathrm{d}y.$$

We apply the change of variables formula to each integral separately. For the integral over $R^+_{u,v}$ we obtain

$$\iint_{R^+_{u,v}} f_{(X,Y)}(x,y)\ \mathrm{d}x\ \mathrm{d}y = \int_{-1}^{v}\int_0^{\sqrt{u}} \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \left|\frac{\partial(x,y)}{\partial(r,w)}\right|\ \mathrm{d}r\ \mathrm{d}w,$$

where

$$\frac{\partial(x,y)}{\partial(r,w)} = \det\begin{pmatrix} w & r \\ \sqrt{1-w^2} & \dfrac{-wr}{\sqrt{1-w^2}} \end{pmatrix} = -\frac{r}{\sqrt{1-w^2}}.$$

It then follows that

$$\iint_{R^+_{u,v}} f_{(X,Y)}(x,y)\;\mathrm{d}x\;\mathrm{d}y = \int_{-1}^{v}\int_0^{\sqrt{u}} \frac{1}{2\pi\sigma^2}\,\frac{r\;e^{-r^2/2\sigma^2}}{\sqrt{1-w^2}}\;\mathrm{d}r\;\mathrm{d}w.$$

A similar calculation for $R^-_{u,v}$ shows that

$$\iint_{R^-_{u,v}} f_{(X,Y)}(x,y)\;\mathrm{d}x\;\mathrm{d}y = \int_{-1}^{v}\int_0^{\sqrt{u}} \frac{1}{2\pi\sigma^2}\,\frac{r\;e^{-r^2/2\sigma^2}}{\sqrt{1-w^2}}\;\mathrm{d}r\;\mathrm{d}w.$$

We then have that

$$\begin{aligned} F_{(U,V)}(u,v) &= \int_{-1}^{v}\int_0^{\sqrt{u}} \frac{1}{\pi\sigma^2}\,\frac{r\;e^{-r^2/2\sigma^2}}{\sqrt{1-w^2}}\;\mathrm{d}r\;\mathrm{d}w \\ &= \int_{-1}^{v} \frac{1}{\pi}\,\frac{1}{\sqrt{1-w^2}}\;\mathrm{d}w \int_0^{\sqrt{u}} \frac{1}{\sigma^2}\;r\;e^{-r^2/2\sigma^2}\;\mathrm{d}r. \end{aligned}$$

Taking partial derivatives with respect to v and w we then obtain the pfd

$$f_{(U,V)}(u,v) = \frac{1}{\pi}\,\frac{1}{\sqrt{1-v^2}}\cdot\frac{1}{\sigma^2}\,\sqrt{u}\;e^{-u/2\sigma^2}\cdot\frac{1}{2\sqrt{u}},$$

where we have used the Fundamental Theorem of Calculus and the Chain Rule. Therefore, the joint pdf for U and V is

$$f_{(U,V)}(u,v) = \frac{1}{\pi}\,\frac{1}{\sqrt{1-v^2}}\cdot\frac{1}{2\sigma^2}\;e^{-u/2\sigma^2} \qquad (2)$$

for $u > 0$ and $-1 < v < 1$, and 0 elsewhere. □

(b) Show that U and V are independent random variables.

Solution: Since the joint pdf of U and V in (2) splits into the product of

$$f_U(u) = \begin{cases} \dfrac{1}{2\sigma^2}\, e^{-u/2\sigma^2} & \text{if } u > 0; \\ \\ 0 & \text{if } u \leqslant 0, \end{cases}$$

and

$$f_V(v) = \begin{cases} \dfrac{1}{\pi}\, \dfrac{1}{\sqrt{1-v^2}} & \text{if } -1 < v < 1; \\ \\ 0 & \text{otherwise,} \end{cases}$$

we see that U and V are independent random variables. Observe that $U \sim \text{exponential}(2\sigma^2)$. □

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf f_X, and let $\overline{X}_n$ denote the sample mean. Prove that the pdf of the sample mean satisfies

$$f_{\overline{X}_n}(t) = n f_Y(nt), \quad \text{for all } t \in \mathbb{R},$$

where $Y = \displaystyle\sum_{i=1}^{n} X_i$.

Solution: First, compute the cdf

$$\begin{aligned} F_{\overline{X}_n}(t) &= \mathrm{P}(\overline{X}_n \leqslant t) \\ &= \mathrm{P}(Y \leqslant nt) \\ &= F_Y(nt), \end{aligned}$$

for $t \in \mathbb{R}$. Differentiate with respect to t to obtain the result. □

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Gamma$(2, \theta)$ distribution, where θ is an unknown parameter. Define $Y = \displaystyle\sum_{i=1}^{n} X_i$.

   (a) Find the distribution of Y and determine c so that the statistic $T = cY$ is an unbiased estimator for θ.

Solution: Compute the mgf of Y to get

$$M_Y(t) \quad = \quad \left(M_{X_1}(t)\right)^n,$$

where

$$M_{X_1}(t) = \left(\frac{1}{1-\theta t}\right)^2 \quad \text{for} \quad t < \frac{1}{\theta}.$$

Thus,

$$M_Y(t) \quad = \quad \left(\frac{1}{1-\theta t}\right)^{2n} \quad \text{for} \quad t < \frac{1}{\theta}, \tag{3}$$

which is the mgf of a Gamma$(2n, \theta)$. Thus, $Y \sim$ Gamma$(2n, \theta)$. We then have that

$$E(Y) = 2n\theta.$$

Consequently,

$$E\left(\frac{Y}{2n}\right) = \theta.$$

It then follows that $T = \dfrac{1}{2n}Y$ is an unbiased estimator for θ. □

(b) If $n = 5$, show that

$$\mathrm{P}\left(9.59 < \frac{2Y}{\theta} < 34.2\right) \approx 0.95.$$

Solution: Let $W = \dfrac{2Y}{\theta}$. It then follows from (3) that

$$M_W(t) \quad = \quad \left(\frac{1}{1-2t}\right)^{2n} \quad \text{for} \quad t < \frac{1}{2},$$

which is the mgf for a $\chi^2(4n)$ distribution. It then follows that W has a χ^2 distribution with $4n$ degrees of freedom. For the case of $n = 5$, we have that $W \sim \chi^2(20)$. Now, for $c < d$, we have that

$$\begin{aligned} \mathrm{P}\left(c < \frac{2Y}{\theta} < d\right) &= \mathrm{P}(c < W < d) \\ &= F_W(d) - F_W(c), \end{aligned}$$

where we have used the fact that W is a continuous random variable. If we want to construct a 95% confidence interval for θ, we need c and d such that

$$\mathrm{P}\left(c < \frac{2Y}{\theta} < d\right) \quad = \quad 0.95,$$

or

$$F_W(d) - F_W(c) = 0.95.$$

We can achieve this by finding c and d such that

$$F_W(d) = 0.975 \quad \text{and} \quad F_W(c) = 0.025.$$

From a table of χ^2 probabilities, or using MS Excel or R, we obtain that

$$c = F_W^{-1}(0.025) \approx 9.59 \quad \text{and} \quad d = F_W^{-1}(0.975) \approx 34.2.$$

we therefore have that

$$\mathrm{P}\left(9.59 < \frac{2Y}{\theta} < 34.2\right) \approx 0.95,$$

which was to be shown. □

(c) Use Part (b) to show that if a sample of size $n = 5$ is collected from a Gamma$(2, \theta)$ distribution, and the sum of the values of the sample is y, then the interval

$$\left(\frac{2y}{34.2}, \frac{2y}{9.59}\right)$$

is a 95% confidence interval for θ.

Solution: Using the result in part be we obtain that

$$\mathrm{P}\left(\frac{1}{34.2} < \frac{\theta}{2Y} < \frac{1}{9.95}\right) \approx 0.95.$$

Consequently,

$$\mathrm{P}\left(\frac{2Y}{34.2} < \theta < \frac{2Y}{9.95}\right) \approx 0.95,$$

and therefore

$$\left(\frac{2Y}{34.2}, \frac{2Y}{9.95}\right)$$

defines a 95% confidence interval for θ. □

(d) Suppose the values in a random sample of size $n = 5$ from a Gamma$(2, \theta)$ distribution are:

44.8079 1.5215 12.1929 12.5734 43.2305

Use the data to obtain a point estimate for θ and a 95% confidence interval for θ.

Give an interpretation of your result.

Solution: In this case the value of Y is the sum of the data values, or $y = 114.3262$. Using the result of part (c), a 95% confidence interval for θ is $(6.69, 23.0)$. This means that if we collect many samples of size $n = 5$ and compute the interval in the manner prescribed in part (c), then, on average, 95% of those intervals will capture the true parameter θ. $\square$

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal(μ, σ^2) distribution and define the statistic

$$T_n = \sum_{i=1}^{n} (X_i - \overline{X}_n)^2,$$

where $\overline{X}_n$ denotes the sample mean. We will show later in this course that $\dfrac{1}{\sigma^2} T_n$ has a χ^2 distribution with $n-1$ degrees of freedom.

(a) Explain how you would use knowledge of the distribution of $\dfrac{1}{\sigma^2} T_n$ to obtain a $100(1-\alpha)\%$ confidence interval for the variance σ^2 of a normal(μ, σ^2) distribution based on a random sample of size n from that distribution.

Solution: Let $Y = \dfrac{1}{\sigma^2} T_n$. Given that $Y \sim \chi^2(n-1)$, where n is known, we can find c and d so that

$$F_{Y}(c) = \frac{\alpha}{2} \quad \text{and} \quad F_{Y}(d) = 1 - \frac{\alpha}{2}.$$

It then follows that

$$\mathrm{P}(c < Y < d) = F_{Y}(d) - F_{Y}(c) = 1 - \alpha,$$

where we have used the fact that Y is a continuous random variable. It then follows that

$$\mathrm{P}\left(c < \frac{1}{\sigma^2} T_n < d\right) = 1 - \alpha,$$

from which we get that

$$\mathrm{P}\left(\frac{1}{d} < \frac{\sigma^2}{T_n} < \frac{1}{c}\right) = 1 - \alpha,$$

or

$$\mathrm{P}\left(\frac{1}{d} T_n < \sigma^2 < \frac{1}{c} T_n\right) = 1 - \alpha.$$

Thus,

$$\left(\frac{1}{d}T_n, \frac{1}{c}T_n\right)$$

is a $100(1-\alpha)\%$ confidence interval for the variance σ^2. □

(b) Give a 90% confidence interval for the variance of a normal(μ, σ^2) distribution based on the statistic T_n, where the sample size, n, is 20.

Solution: Here, $\alpha = 0.1$ and $Y \sim \chi^2(19)$. Therefore,

$$c = F_{_Y}^{-1}(0.05) = 8.91 \quad \text{and} \quad d = F_{_Y}^{-1}(0.95) = 30.1.$$

we then have that a 90% confidence interval for the variance in this case is

$$\left(\frac{1}{30.1}T_n, \frac{1}{8.91}T_n\right).$$

□

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with unknown expectation, μ, and unknown variance, σ^2. Define the statistic

$$T_n = \sum_{i=1}^{n}(X_i - \overline{X}_n)^2,$$

where $\overline{X}_n$ denotes the sample mean.

(a) Starting with

$$(X_i - \mu)^2 = [(X_i - \overline{X}_n) + (\overline{X}_n - \mu)]^2,$$

where $\overline{X}_n$ denotes the sample mean, derive the identity

$$\sum_{i=1}^{n}(X_i - \mu)^2 = T_n + n(\overline{X}_n - \mu)^2. \tag{4}$$

Solution: Compute

$$(X_i - \mu)^2 = (X_i - \overline{X}_n)^2 + 2(\overline{X}_n - \mu)(X_i - \overline{X}_n) + (\overline{X}_n - \mu)^2,$$

then add

$$\begin{aligned}\sum_{i=1}^{n}(X_i - \mu)^2 &= \sum_{i=1}^{n}(X_i - \overline{X}_n)^2 + 2(\overline{X}_n - \mu)\sum_{i=1}^{n}(X_i - \overline{X}_n) \\ &\quad + \sum_{i=1}^{n}(\overline{X}_n - \mu)^2,\end{aligned}$$

where

$$\sum_{i=1}^{n}(X_i - \overline{X}_n) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X}_n = n\overline{X}_n - n\overline{X}_n = 0,$$

and

$$\sum_{i=1}^{n}(\overline{X}_n - \mu)^2 = n(\overline{X}_n - \mu)^2.$$

Thus,

$$\sum_{i=1}^{n}(X_i - \mu)^2 \quad = \quad \sum_{i=1}^{n}(X_i - \overline{X}_n)^2 + n(\overline{X}_n - \mu)^2,$$

which is (4). □

(b) Take expectations on both sides of equation (4) to derive a formula for $E(T_n)$ in terms of σ^2. Is T_n an unbiased estimator for σ^2?

***Solution**:* Taking expectation on both sided of (4) we have

$$\sum_{i=1}^{n} E(X_i - \mu)^2 \quad = \quad E(T_n) + nE(\overline{X}_n - \mu)^2,$$

where we have used the linearity of the expectation operator, E. Thus,

$$\sum_{i=1}^{n} \text{var}(X_i) \quad = \quad E(T_n) + n \ \text{var}(\overline{X}_n),$$

where

$$\text{var}(X_i) = \sigma^2 \quad \text{for all } i = 1, 2, \ldots, n,$$

and

$$\text{var}(\overline{X}_n) = \frac{\sigma^2}{n}.$$

Consequently,

$$n\sigma^2 = E(T_n) + n \ \frac{\sigma^2}{n},$$

from which we get that

$$E(T_n) = (n-1) \ \sigma^2.$$

Hence, T_n is not an unbiased estimator for σ^2. □