Assignment #5

Due on Wednesday, October 26, 2011

Read Section 4.3 on *Differentiability,* pp. 189–195, in Baxandall and Liebek's text.
Read Section 3.3 on *Linear Approximation and Differentiability,* pp. 113–123, in Baxandall and Liebek's text.

Read Section 4.1 on *Definition of Differentiability* in the class Lecture Notes (pp. 41–43).

Read Section 4.2 on *The Derivative* in the class Lecture Notes (pp. 43–44).

Read Section 4.3 on *Differentiable Scalar Fields* in the class Lecture Notes (pp. 44–49).

Do the following problems

1. Let f denote a real valued function defined on some open interval around $a \in \mathbb{R}$. Consider a line of slope m and equation

$$L(x) = f(a) + m(x - a) \quad \text{for all} \quad x \in \mathbb{R}.$$

   Suppose that this line if the best approximation to the function f at a in the sense that

$$\lim_{x \to a} \frac{|E(x)|}{|x - a|} = 0,$$

   where $E(x) = f(x) - L(x)$ for all x in the interval in which f is defined.

   Prove that f is differentiable at a, and that $f'(a) = m$.

2. Prove that if F is differentiable at u, then it is also continuous at u.

   Give an example that shows that the converse of this assertion is not true

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sqrt{|xy|}$ for all $(x, y) \in \mathbb{R}^2$. Show that f is not differentiable at $(0, 0)$.

4. Is $f(x, y, z) = x\sqrt{y^2 + z^2}$ differentiable at $(0, 0, 0)$? Prove your assertion.

5. Is the scalar field

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

continuous at the origin? Is it differentiable at the origin?

6. Let $U = \mathbb{R}^n \backslash \{\mathbf{0}\} = \{v \in \mathbb{R}^n \mid v \neq \mathbf{0}\}$ and define $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \|v\|$ for all $v \in \mathbb{R}$.

   (a) Prove that f is differentiable on U.

   (b) Prove that f is not differentiable at the origin in $\mathbb{R}^n$.

7. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be give by $f(x,y,z) = x^2y + y^2z + z^2x$, for all $(x,y,z) \in \mathbb{R}^3$. Compute all the first partial derivatives of f and verify that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 3f.$$

8. Find the gradient of f for each of the following scalar fields:

   (a) $f(x,y,z) = xe^{yz}$,

   (b) $f(x,y,z) = 1/\sqrt{x^2+y^2+z^2}$, $(x,y,z) \neq (0,0,0)$.

9. Let $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right), & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

   (a) Show that the partial derivatives of f with respect to x and y do exist at $(0,0)$, and compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

   (b) Show that the partial derivatives of f with respect to x and y are not continuous at $(0,0)$.

10. Let f be as in the previous problem. Show that f is differentiable at $(0,0)$, and compute $Df(0,0)$.