Assignment #6

Due on Wednesday, November 2, 2011

Read Section 3.6 on *The Chain Rule and the Rate of Change along a Path,* pp. 133–136, in Baxandall and Liebek's text.

Read Section 3.7 on *Directional Derivatives,* pp. 138–141, in Baxandall and Liebek's text.

Read Section 3.8 on *The Gradient and Smooth Surfaces,* pp. 142–151, in Baxandall and Liebek's text.

Read Section 4.4 on *The Chain Rule,* pp. 197–202, in Baxandall and Liebek's text.

Read Section 4.6 on *Derivatives of Compositions* in the class Lecture Notes (pp. 56–60).

Do the following problems

1. Let $U = \{(x,y)\in\mathbb{R}^2 \mid x\neq 0\}$ and define $f\colon U\to\mathbb{R}$ by $f(x,y) = \arctan\left(\dfrac{y}{x}\right)$, for all $(x,y)\in U$.

   (a) Compute the gradient of f in U.

   (b) Let I be an open interval and $\sigma\colon I\to U$ be a differentiable path given by $\sigma(t) = (x(t), y(t))$ for $t\in I$. Define $\theta\colon I\to\mathbb{R}$ by $\theta(t) = (f\circ\sigma)(t)$ for all $t\in I$. Apply the Chain Rule to verify that $\theta' = \dfrac{-yx'+xy'}{x^2+y^2}$.

   (c) Apply the result from part (b) to the path $\sigma\colon \left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\to\mathbb{R}^2$ given by $\sigma(t) = (\cos t, \sin t)$, for $t\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

2. Suppose that the temperature in a region of space is given by a function $T\colon\mathbb{R}^3\to\mathbb{R}$ given by

$$T(x,y,z) = kx^2(y-z), \qquad \text{for all } (x,y,z)\in\mathbb{R}^3,$$

   and some positive constant k.

   An insect flies in the region along a path modeled by a C^1 function $\sigma\colon\mathbb{R}\to\mathbb{R}^3$. Suppose that at time $t=0$ the insect is located at $(0,0,0)$ and its velocity is $\sigma'(0) = \widehat{i}+\widehat{j}+2\widehat{k}$. Compute the rate of change of temperature sensed by the insect at time $t=0$.

3. Let I be an open interval of real numbers and U be an open subset of $\mathbb{R}^n$. Suppose that $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path and that $f\colon U \to \mathbb{R}$ is a differentiable scalar field. Assume also that the image of I under σ, $\sigma(I)$, is contained in U. Suppose also that the derivative of the path σ satisfies

$$\sigma'(t) = -\nabla f(\sigma(t)) \quad \text{for all} \quad t \in I.$$

Show that if the gradient of f along the path σ is never zero, then f decreases along the path as t increases.

Suggestion: Use the Chain Rule to compute the derivative of $f(\sigma(t))$.

4. A set $U \subseteq \mathbb{R}^n$ is said to be **path connected** iff for any vectors x and y in U, there exists a differentiable path $\sigma\colon [0,1] \to \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0,1]$; i.e., any two elements in U can be connected by a differentiable path whose image is entirely contained in U.

Suppose that U is an open, path connected subset of $\mathbb{R}^n$. Let $f\colon U \to \mathbb{R}$ be a differentiable scalar field such that $\nabla f(x)$ is the zero vector for all $x \in U$. Prove that f must be constant.

5. Let U be an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ be a differentiable scalar filed defined on U. The function f is said to be homogeneous of order k if

$$f(tv) = t^k f(v),$$

for all $v \in U$ and all positive $t \in \mathbb{R}$ such that $tv \in U$.

(a) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy$, for all $(x,y) \in \mathbb{R}^2$, is homogeneous of order 2.

(b) Give examples of a scalar field which is homogenous of order 1 and of a scalar field which is homogeneous of order 0.

(c) Prove Euler's Theorem: If $f\colon U \to \mathbb{R}$ is differentiable and homogenous of order k, then

$$x_1\frac{\partial f}{\partial x_1}(x) + x_2\frac{\partial f}{\partial x_2}(x) + \cdots + x_n\frac{\partial f}{\partial x_n}(x) = kf(x),$$

for all $x = (x_1, x_2, \ldots, x_n) \in U$.

Suggestion: Let $\sigma\colon \mathbb{R} \to \mathbb{R}^n$ be given by $\sigma(t) = tx$, for all $t \in \mathbb{R}$ and $x = (x_1, x_2, \ldots, x_n) \in U$, and apply the Chain Rule to the composition $f \circ \sigma$.

6. Let x and y be functions of u and v: $x = x(u,v)$, $y = y(u,v)$, and let $f(x,y)$ be a scalar field. Find $\partial f/\partial u$ and $\partial f/\partial v$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$.

7. For f, x and y as in Problem 6, express $\dfrac{\partial^2 f}{\partial u^2}$ in terms of the partial derivatives of f with respect to x and y and the partial derivatives of x and y with respect to u. Assume that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

8. Let $G\colon \mathbb{R}^n \to \mathbb{R}^m$ and $F\colon \mathbb{R}^m \to \mathbb{R}^n$ be differentiable functions such that
$$(F \circ G)(x) = x, \qquad \text{for all } x \in \mathbb{R}^n.$$
Put $y = G(x)$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. Apply the Chain Rule to show that
$$\frac{\partial f_i}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial f_i}{\partial y_2}\frac{\partial y_2}{\partial x_j} + \cdots + \frac{\partial f_i}{\partial y_m}\frac{\partial y_m}{\partial x_j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j, \end{cases}$$
where $f_1, f_2, \ldots, f_n\colon \mathbb{R}^m \to \mathbb{R}$ are the components of the vector field F.

9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^2 + y^2 + xy$, for all $(x,y) \in \mathbb{R}^2$, and assume that $x = r\cos\theta$ and $y = r\sin\theta$ for $r \geqslant 0$ and $\theta \in \mathbb{R}$. Put $z = f(x,y)$ for all $(x,y) \in \mathbb{R}^2$. Use the Chain Rule to compute $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$

10. Let f be a scalar field defined on (x,y) where $x = r\cos\theta$, $y = r\sin\theta$. Show that
$$\nabla f = \frac{\partial f}{\partial r}\overrightarrow{\mathbf{u}_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\overrightarrow{\mathbf{u}_\theta},$$
where $\overrightarrow{\mathbf{u}_r} = (\cos\theta, \sin\theta)$ and $\overrightarrow{\mathbf{u}_\theta} = (-\sin\theta, \cos\theta)$.

Hint: First find $\partial f/\partial r$ and $\partial f/\partial \theta$ in terms of $\partial f/\partial x$ and $\partial f/\partial y$ and then solve for $\partial f/\partial x$ and $\partial f/\partial y$ int terms of $\partial f/\partial r$ and $\partial f/\partial \theta$.