Assignment #7

Due on Wednesday, November 9, 2011

Read Section 3.6 on *The Chain Rule and the Rate of Change along a Path,* pp. 133–136, in Baxandall and Liebek's text.

Read Section 3.7 on *Directional Derivatives,* pp. 138–141, in Baxandall and Liebek's text.

Read Section 3.8 on *The Gradient and Smooth Surfaces,* pp. 142–151, in Baxandall and Liebek's text.

Read Section 4.4 on *The Chain Rule,* pp. 197–202, in Baxandall and Liebek's text.

Read Section 4.6 on *Derivatives of Compositions* in the class Lecture Notes (pp. 56–60).

Do the following problems

1. Let U denote an open subset of $\mathbb{R}^2$ and $F\colon U \to \mathbb{R}^2$ denote a vector field given by
$$F(x,y) = f(x,y)\widehat{i} + g(x,y)\widehat{j}, \quad \text{for all } (x,y) \in U,$$
where f and g are differentiable scalar fields defined in U. We define the divergence of F, denoted by $\text{div}F$, to be a scalar field, $\text{div}F\colon U \to \mathbb{R}$, given by
$$\text{div}F(x,y) = \frac{\partial f}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y), \quad \text{for all } (x,y) \in U. \tag{1}$$
In some texts, $\text{div}F$ is denoted by $\nabla \cdot F$; that is, $\text{div}F$ is, formally, the dot product of the vector differential operator
$$\nabla = \frac{\partial}{\partial x}\,\widehat{i} + \frac{\partial}{\partial y}\,\widehat{j}$$
with the vector field $F = f\,\widehat{i} + g\,\widehat{j}$.

   Use the definition of divergence in (1) to compute the divergence of the following vector fields

   (a) $F(x,y) = \frac{1}{3}x^3\,\widehat{i} + \frac{1}{3}y^3\,\widehat{j}$ for $(x,y) \in \mathbb{R}^2$.

   (b) $F(x,y) = (x^2 - y^2)\,\widehat{i} + 2xy\,\widehat{j}$ for $(x,y) \in \mathbb{R}^2$.

2. Let U denote an open subset of $\mathbb{R}^2$ and $f\colon U \to \mathbb{R}$ be a scalar field whose second partial derivatives exist in U. Use the definition of ∇f and of divergence in (1) to show that

$$\text{div}\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \tag{2}$$

The expression on the right–hand side of the equation in (2) is called the Laplacian of f and is usually denoted by Δf or $\nabla^2 f$. We then have that, for a scalar field in U with second partial derivatives,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \quad \text{or} \quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \quad \text{in } U.$$

3. Let $U = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \neq 0$ and put $r = \sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Let $g\colon (0,\infty) \to \mathbb{R}$ denote a twice–differentiable real valued function. Define f to be the composition of g with the function $r\colon \mathbb{R}^2 \to \mathbb{R}$; in other words,

$$f(x,y) = g(r), \quad \text{where } r = \sqrt{x^2+y^2}, \quad \text{for all } (x,y) \in U.$$

   (a) Use the Chain Rule to compute ∇f in U, and express it in terms of r, $g'(r)$ and the vector $\overrightarrow{r} = x\,\hat{\imath} + y\,\hat{\jmath}$.

   (b) Compute the Laplacian, Δf or $\nabla^2 f$, of f in U, and express it in terms of r, $g'(r)$ and $g''(r)$.

4. Let U be as in Problem 3. Put $g(t) = \ln t$ for all $t > 0$ and let

$$f(x,y) = \ln\sqrt{x^2+y^2}, \quad \text{for } (x,y) \in U.$$

Use your result from Problem 3 to compute ∇f and Δf in U. What do you conclude about the Laplacian of f in U?

5. Let U denote an open subset of $\mathbb{R}^2$ and $f\colon U \to \mathbb{R}$ and $g\colon U \to \mathbb{R}$ be differentiable scalar fields in U. Assume that the second partial derivatives of g exist in U. Derive the identity

$$\text{div}(f\nabla g) = \nabla f \cdot \nabla g + f\Delta g,$$

where Δg denotes the Laplacian of g in U.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,t) = f(x - ct) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2,$$

where c is a real constant. Show that $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

7. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.

   (a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.

   (b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.

   (c) Find largest and the smallest value of f on C.

8. Let D denote an open region in $\mathbb{R}^2$ and $f\colon D \to \mathbb{R}$ denote a scalar field whose second partial derivatives exist in D. Fix $(x,y) \in D$, and define the scalar map

$$S(h,k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y),$$

where $|h|$ and $|k|$ are sufficiently small.

   (a) Apply the Mean Value Theorem to obtain an $\overline{x}$ in the interval $(x, x+h)$, or $(x+h, x)$ (depending on whether h is positive or negative, respectively) such that $S(h,k) = \left(\dfrac{\partial f}{\partial x}(\overline{x}, y+k) - \dfrac{\partial f}{\partial x}(\overline{x}, y) \right) h$.

   (b) Apply the Mean Value Theorem to obtain a $\overline{y}$ in the interval $(y, y+k)$, or $(y+k, y)$ (depending on whether k is positive or negative, respectively) such that $S(h,k) = \dfrac{\partial^2 f}{\partial y \partial x}(\overline{x}, \overline{y}) hk$.

9. *(Continuation of Problem 8.)*

   (c) The function f is said to be of class C^2 if all its second partial derivatives are continuous on D.

   Show that if f is of class C^2, then $\displaystyle\lim_{(h,k)\to(0,0)} \frac{S(h,k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(x,y)$.

   (d) Deduce that if f is of class C^2, then

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y);$$

   that is, the *mixed* second partial derivatives are the same for C^2 maps.

10. Let Ω denote an open region in three–dimensional Euclidean space, $\mathbb{R}^3$. Let I denote an open interval. The set

$$\Omega \times I = \{(x, y, z, t) \in \mathbb{R}^4 \mid (x, y, z) \in \Omega \text{ and } t \in I\}$$

is called the Cartesian product of Ω and I. Let $V\colon \Omega \times I \to \mathbb{R}^3$ denote a vector field defined in Ω which also depends on time $t \in I$. For instance, $V(x, y, z, t)$ denotes the velocity of a fluid element located at $(x, y, z) \in \Omega$ at time t.

Let $\sigma\colon I \to \mathbb{R}^3$ denote a differentiable path in $\mathbb{R}^3$ such that $\sigma(t) \in \Omega$ for all $t \in I$. Furthermore, assume that

$$\sigma'(t) = V(\sigma(t), t), \qquad \text{for all } t \in I;$$

that is, the path σ is always tangent to the field V at every point $\sigma(t)$ and time t.

Apply the Chain Rule to verify that, for any differentiable scalar field $f\colon \Omega \times I \to \mathbb{R}$

$$\frac{d}{dt}[f(\sigma(t), t)] = \frac{\partial f}{\partial t} + V \cdot \nabla f(\sigma(t), t), \qquad \text{for all } t \in \mathbb{R}.$$

Suggestion: Write $\sigma(t) = (x(t), y(t), z(t))$, for all $t \in I$, where x, y and z are differentiable functions of $t \in I$.