Assignment #8

Due on Wednesday, November 16, 2011

Read Section 2.6 on *Curves and Simple Arcs and Orientation,* pp. 45–56, in Baxandall and Liebek's text.

Read Section 2.7 on *Path Length and Length of Simple Arcs,* pp. 59–66, in Baxandall and Liebek's text.

Read Section 5.2 on *Integral of a scalar Field Along a Path,* pp. 269–279, in Baxandall and Liebek's text.

Read Section 5.3 on *Integral of a Vector Field Along a Path,* pp. 281–290, in Baxandall and Liebek's text.

Read Section 5.1 on the *Path Integral* in the class Lecture Notes (pp. 61–68).

Read Section 5.2 on *Line Integrals* in the class Lecture Notes (pp. 69–72).

Background and Definitions

- *(Parametrization)* Let I denote and interval of real numbers, $\sigma\colon I \to \mathbb{R}^n$ be a continuous path, and let C denote the image of I under σ. Then, C is called a curve in $\mathbb{R}^n$. If σ is one–to–one on I, then σ is called a parametrization of C. For example, if v and u are distinct vectors in $\mathbb{R}^n$, then

  $$\sigma(t) = u + t(v - u), \quad \text{for } 0 \leqslant t \leqslant 1,$$

  is a parametrization of the straight line segment from the point u to the point v in $\mathbb{R}^n$.

- *(C^1 Curves)* If C is parametrized by a C^1 path, $\sigma\colon I \to \mathbb{R}^n$, with $\sigma'(t) \neq \mathbf{0}$ for all $t \in I$, the curve C is said to be a C^1 curve or a smooth curve.

- *(Simple Closed Curves)* If $\sigma\colon [a, b] \to \mathbb{R}^n$ is a parametrization of a curve C, with $\sigma(a) = \sigma(b)$ and $\sigma\colon [a, b) \to \mathbb{R}^n$ being one–to–one, then C is said to be a simple closed curve.

- *(Reparametrizations)* Let $\sigma\colon [a, b] \to \mathbb{R}^n$ be a differentiable, one–to–one path. Suppose also that $\sigma'(t)$, is never the zero vector. Let $h\colon [c, d] \to [a, b]$ be a differentiable, one–to–one and onto map such that $h'(t) \neq 0$ for all $t \in [c, d]$. Define

  $$\gamma(t) = \sigma(h(t)) \quad \text{for all } \ t \in [c, d].$$

  $\gamma\colon [c, d] \to \mathbb{R}^n$ is a called a *reparametrization* of σ

- *(Arc Length Parameter)* Let I denote an open interval in $\mathbb{R}$, and $\sigma\colon I \to \mathbb{R}^n$ be a parametrization of a curve C. For fixed $a \in I$, define

$$s(t) = \int_a^t \|\sigma'(\tau)\| \, \mathrm{d}\tau \quad \text{for all} \;\; t \in I. \tag{1}$$

  The parameter $s = s(t)$ measures the length along the curve C from the point $\sigma(a)$ to the point $\sigma(t)$.

- Let U be an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ be a continuous scalar field. Let $C \subset U$ be a C^1 simple curve. We define the integral of f over C, denoted $\displaystyle\int_C f \, \mathrm{d}s$, to be

$$\int_C f \, \mathrm{d}s = \int_a^b f(\sigma(t))\|\sigma'(t)\| \, \mathrm{d}t,$$

  where $\sigma\colon [a,b] \to \mathbb{R}^n$ is any C^1 parametrization of C.

- A curve, C, is said to be piece–wise C^1 if C can be decomposed into a finite union of C^1 simple curves, $C_1, C_2, \ldots, C_k$:

$$C = \bigcup_{i=1}^k C_i.$$

  If $C \subset U$, where U is an open subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is a continuous scalar field, we define the integral of f over C by

$$\int_C f \, \mathrm{d}s = \sum_{i=i}^k \int_{C_i} f \, \mathrm{d}s.$$

- *(Flux Across a Simple, Closed Curve in $\mathbb{R}^2$)* Let U denote an open subset of $\mathbb{R}^2$ and $F\colon U \to \mathbb{R}^2$ be a two–dimensional vector field given by

$$F(x,y) = P(x,y)\,\widehat{i} + Q(x,y)\,\widehat{j}, \qquad \text{for all } (x,y) \in U,$$

  where P and Q are scalar fields defined in U. Let C denote a simple, piece–wise C^1, closed curve contained in U, which is oriented in the counterclockwise sense. The flux of F across C, denoted by $\displaystyle\oint_C F \cdot \widehat{n} \;\; \mathrm{d}s$, is defined by

$$\oint_C F \cdot \widehat{n} \;\; \mathrm{d}s = \int_C P(x,y) \;\; \mathrm{d}y - Q(x,y) \;\; \mathrm{d}x,$$

  where $\widehat{n}$ denotes the outward unit normal to the curve C, wherever it is defined.

Do the following problems

1. Let $\sigma(t) = (x(t), y(t))$, for $t \in [a,b]$, be a parametrization of a simple closed curve. Assume that σ is oriented in the counterclockwise sense. Give the unit vector to the curve at $\sigma(t)$, for $t \in (a,b)$, which is perpendicular to $\sigma'(t)$ and points towards the exterior of the curve.

2. Show that the arc length parameter defined in (1) is differentiable on I and compute $s'(t)$ for all $t \in I$. Deduce that $s(t)$ is a strictly increasing function of t in I.

3. Find the mass of a wire that is parametrized by
$$C = \left\{ \left(\frac{3}{2}t^2, (1+2t)^{3/2} \right) \;\Big|\; 0 \leqslant t \leqslant 2 \right\}$$
and has a density given by $\rho(x,y) = 2x + 1$.

4. Consider a portion of a helix, C, parametrized by the path
$$\sigma(t) = (\cos t, t, \sin t) \quad \text{for} \quad 0 \leqslant t \leqslant \pi.$$
Let $f(x,y,z) = x^2 + y^2 + z^2$ for all $(x,y,z) \in \mathbb{R}^3$. Evaluate $\displaystyle\int_C f$.

5. Evaluate $\displaystyle\int_C (x^3 - yz)\, \mathrm{d}s$, where C is the intersection of the planes $x + y - z = 1$ and $z = 3x$ from $x = 0$ to $x = 1$.

6. Let C denote the boundary of the square
$$R = \{(x,y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}.$$
Evaluate the integral of $f(x,y) = xy^2$, for $(x,y) \in \mathbb{R}^2$, over C.

   Note: Observe that C is not a C^1 curve, but it can be decomposed into an union of four simple, C^1 curves.

7. Consider a portion of a helix, C, parametrized by the path

$$\sigma(t) = (\cos t, t, \sin t) \quad \text{for} \quad 0 \leqslant t \leqslant \pi.$$

   Let $F(x,y,z) = x\,\widehat{i} + y\,\widehat{j} + z\,\widehat{k}$, for all $(x,y,z) \in \mathbb{R}^3$, be a vector field in $\mathbb{R}^3$. Evaluate the line integral $\int_C F \cdot \, \mathrm{d}\overrightarrow{r}$; that is, the integral of the tangential component of the field F along the curve C.

8. Let $f\colon U \to \mathbb{R}$ be a C^1 scalar field defined on an open subset U of $\mathbb{R}^n$. Define the vector field $F\colon U \to \mathbb{R}^n$ by $F(x) = \nabla f(x)$ for all $x \in U$. Suppose that C is a C^1 simple curve in U connecting the point x to the point y in U. Show that

$$\int_C F \cdot \, \mathrm{d}\overrightarrow{r} = f(y) - f(x).$$

   Conclude therefore that the line integral of F along a path from x to y in U is independent of the path connecting x to y. The field F is called a *gradient field.*

9. Let $F(x,y) = x^2\,\widehat{i} + y^2\,\widehat{j}$ and C be the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, oriented in the in the counterclockwise sense. Compute the flux of F across C.

10. Compute the flux, $\oint_C F \cdot \widehat{n} \, \mathrm{d}s$, where $F(x,y) = x\,\widehat{i} + y\,\widehat{j}$, for all $(x,y) \in \mathbb{R}^2$ and C is the unit circle oriented in the counterclockwise sense.