Solutions to Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by
$$4x - y - 3z = 12.$$

***Solution**:* The point $P_o(3, 0, 0)$ is in the plane. Let
$$w = \overrightarrow{P_oP} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$$

The vector $n = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ is orthogonal to the plane. To find the shortest distance, d, from P to the plane, we compute the norm of the orthogonal projection of w onto n; that is,
$$d = \|\mathrm{P}_{\widehat{n}}(w)\|,$$
where
$$\widehat{n} = \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix},$$
a unit vector in the direction of n, and
$$\mathrm{P}_{\widehat{n}}(w) = (w \cdot \widehat{n})\widehat{n}.$$

It then follows that
$$d = |w \cdot \widehat{n}|,$$
where $w \cdot \widehat{n} = \dfrac{1}{\sqrt{26}}(4 + 21) = \dfrac{25}{\sqrt{26}}$. Hence, $d = \dfrac{25\sqrt{26}}{26} \approx 4.9.$ □

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\left\{ \begin{array}{rcl} x & = & -1 + 4t, \\ y & = & -7t, \\ z & = & 2 - t. \end{array} \right.$$

Solution: The point $P_o(-1, 0, 2)$ is on the line. The vector

$$v = \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix}$$

gives the direction of the line. Put

$$w = \overrightarrow{P_oP} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}.$$

The vectors v and w determine a parallelogram whose area is the norm of v times the shortest distance, d, from P to the line determined by v at P_o. We then have that

$$\text{area}(P(v, w)) = \|v\|d,$$

from which we get that

$$d = \frac{\text{area}(P(v, w))}{\|v\|}.$$

On the other hand,

$$\text{area}(P(v, w)) = \|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 4 & -7 & -1 \\ 5 & 0 & -9 \end{vmatrix} = 63\widehat{i} + 31\widehat{j} + 35\widehat{k}.$$

Thus, $\|v \times w\| = \sqrt{(63)^2 + (31)^2 + (35)^2} = \sqrt{6155}$ and therefore

$$d = \frac{\sqrt{6155}}{\sqrt{66}} \approx 9.7.$$

$\square$

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1, 1, 0)$, $(2, 0, 1)$ and $(0, 3, 1)$

***Solution**:* Label the points $P_o(1,1,0)$, $P_1(2,0,1)$ and $P_2(0,3,1)$ and define the vectors

$$v = \overrightarrow{P_oP_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w = \overrightarrow{P_oP_2} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

The area of the triangle determined by the points P_o, P_1 and P_2 is then half of the area of the parallelogram determined by the vectors v and w. Thus,

$$\text{area}(\triangle P_oP_1P_2) = \frac{1}{2}\|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -3\widehat{i} - 2\widehat{j} + \widehat{k}.$$

Consequently, $\text{area}(\triangle P_oP_1P_2) = \dfrac{1}{2}\sqrt{9+4+1} = \dfrac{\sqrt{14}}{2} \approx 1.87.$ $\square$

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

***Solution**:* The area of the parallelogram determined by v and $w+\lambda v$ is

$$\text{area}(P(v, w + \lambda v)) = \|v \times (w + \lambda v)\|,$$

where

$$v \times (w + \lambda v) = v \times w + \lambda v \times v = v \times w.$$

Consequently, $\text{area}(P(v, w + \lambda v)) = \|v \times w\| = \text{area}(P(v, w)).$ $\square$

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map

$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n$$

is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Solution: $P_{\widehat{u}}(v) = (v \cdot \widehat{u})\widehat{u}$ for all $v \in \mathbb{R}^n$. Consequently, for any $w, v \in \mathbb{R}^n$,

$$\begin{aligned} P_{\widehat{u}}(w) - P_{\widehat{u}}(v) &= (w \cdot \widehat{u})\widehat{u} - (v \cdot \widehat{u})\widehat{u} \\ &= (w \cdot \widehat{u} - v \cdot \widehat{u})\widehat{u} \\ &= [(w - v) \cdot \widehat{u}]\widehat{u}. \end{aligned}$$

It then follows that

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = |(w - v) \cdot \widehat{u}|,$$

since $\|\widehat{u}\| = 1$. Hence, by the Cauchy–Schwarz inequality,

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| \leqslant \|w - v\|.$$

Applying the Squeeze Theorem we then get that

$$\lim_{\|w-v\|\to 0} \|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = 0,$$

which shows that $P_{\widehat{u}}$ is continuous at every $v \in V$. □

6. Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is continuous at $(0,0)$.

Solution: For $(x,y) \neq (0,0)$

$$\begin{aligned} |f(x,y)| &= \frac{x^2|y|}{x^2 + y^2} \\ &\leqslant |y|, \end{aligned}$$

since $x^2 \leqslant x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. We then have that, for $(x,y) \neq (0,0)$,

$$|f(x,y)| \leqslant \sqrt{x^2 + y^2},$$

which implies that

$$0 \leqslant |f(x,y) - f(0,0)| \leqslant \|(x,y) - (0,0)\|,$$

for $(x, y) \neq (0, 0)$. Thus, by the Squeeze Theorem,

$$\lim_{\|(x,y)-(0,0)\|\to 0} |f(x, y) - f(0, 0)| = 0,$$

which shows that f is continuous at $(0, 0)$. $\square$

7. Show that

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$.

***Solution**:* Let $\sigma_1(t) = (t, t)$ for all $t \in \mathbb{R}$ and observe that

$$\lim_{t\to 0} \sigma_1(t) = (0, 0)$$

and

$$f(\sigma(t)) = 0, \quad \text{for all } t \neq 0.$$

It then follows that

$$\lim_{t\to 0} f(\sigma_1(t)) = 0.$$

Thus, if f were continuous at $(0, 0)$, we would have that

$$f(0, 0) = 0. \tag{1}$$

On the other hand, if we let $\sigma_2(t) = (t, 0)$, we would have that

$$\lim_{t\to 0} \sigma_2(t) = (0, 0)$$

and

$$f(\sigma(t)) = 1, \quad \text{for all } t \neq 0.$$

Thus, if f were continuous at $(0, 0)$, we would have that

$$f(0, 0) = 1,$$

which is in contradiction with (1). This contradiction shows that f is not continuous at $(0, 0)$. $\square$

8. Determine the value of L that would make the function

$$f(x,y) = \begin{cases} x \ \sin\left(\dfrac{1}{y}\right) & \text{if } \ y \neq 0; \\ \\ L & \text{otherwise}\ , \end{cases}$$

continuous at $(0,0)$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous on $\mathbb{R}^2$? Justify your answer.

Solution: Observe that, for $y \neq 0$,

$$\begin{aligned} |f(x,y)| &= \left| x \ \sin\left(\frac{1}{y}\right) \right| \\ &= |x| \ \left| \sin\left(\frac{1}{y}\right) \right| \\ &\leqslant |x| \\ &\leqslant \sqrt{x^2+y^2}. \end{aligned}$$

It then follows that, for $y \neq 0$,

$$0 \leqslant |f(x,y)| \leqslant \|(x,y)\|.$$

Consequently, by the Squeeze Theorem,

$$\lim_{\|(x,y)\|\to 0} |f(x,y)| = 0.$$

This suggests that we define $L = 0$. If this is the case,

$$\lim_{\|(x,y)\|\to 0} |f(x,y) - f(0,0)| = 0,$$

which shows that f is continuous at $(0,0)$ if $L = 0$.

Next, assume now that $L = 0$ in the definition of f. Then, for any $a \neq 0$, f fails for be continuous at $(a,0)$. To see why this is case, note that for any $y \neq 0$

$$f(a,y) = a \ \sin\left(\frac{1}{y}\right)$$

and the limit of $\sin\left(\dfrac{1}{y}\right)$ as $y \to 0$ does not exist. □

9. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ be the path given by

$$\sigma(t) = (2\cos t, \sin t), \qquad \text{for } t \in \mathbb{R}.$$

(a) Sketch the image of σ.

Solution: The image of σ is the set,

$$\sigma(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 \mid x = 2\cos t, y = \sin t, \text{ for } t \in \mathbb{R}\},$$

of points, (x, y), in $\mathbb{R}^2$ such that

$$x = 2\cos t \quad \text{ and } \quad y = \sin t, \tag{2}$$

for $t \in \mathbb{R}$. It follows from the equations in (3) that $(x, y) \in \sigma(\mathbb{R})$ if and only if

$$\frac{x^2}{4} + y^2 = 1,$$

which shows that $\sigma(\mathbb{R})$ is the ellipse pictured in Figure 1 $\square$

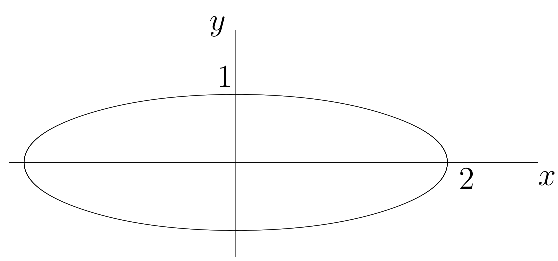

Figure 1: Sketch of ellipse

(b) Find a tangent vector to the path at $t = \pi/4$.

Solution: Compute the derivative of the path

$$\sigma'(t) = (-2\sin t, \cos t), \qquad \text{for all } t \in \mathbb{R}.$$

Then, a tangent vector to the path at $t = \pi/4$ is

$$v = \sigma'(\pi/4) = \left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right). \tag{3}$$

$\square$

(c) Give the parametric equations to the tangent line to the path at $t = \pi/4$. Sketch the line.

Solution: The point on the path at time $t = \pi/4$ has coordinates

$$\sigma(\pi/4) = \left(\sqrt{2}, \frac{\sqrt{2}}{2}\right).$$

The line through this point in the direction of the vector v in (3) has parametric equations

$$\begin{cases} x = \sqrt{2} - \sqrt{2}\left(t - \dfrac{\pi}{4}\right); \\ \\ y = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(t - \dfrac{\pi}{4}\right), \end{cases}$$

for $t \in \mathbb{R}$.
Sketch of the tangent line to the path σ and $\sigma(\pi/4)$ is shown in Figure Figure 2 $\square$

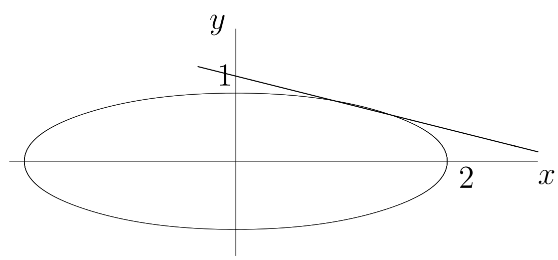

Figure 2: Tangent Line at $\sigma(\pi/4)$

10. Let I denote an open interval, and $\sigma\colon I \to \mathbb{R}^n$ and $\gamma\colon I \to \mathbb{R}^n$ be differentiable paths on I. Define $h(t) = \sigma(t) \cdot \gamma(t)$ for all $t \in \mathbb{R}$. Show that $h\colon I \to \mathbb{R}$ is differentiable on I and verify that

$$h'(t) = \sigma'(t) \cdot \gamma(t) + \sigma(t) \cdot \gamma'(t), \qquad \text{for all } t \in I. \tag{4}$$

Solution: Write $\sigma(t) = (x_1(t), x_2(t), \ldots, x_n(t)$ and $\gamma(t) = (y_1(t), y_2(t), \ldots, y_n(t)$, for all $t \in I$, where $x_i\colon I \to \mathbb{R}$ and $y_i\colon I \to \mathbb{R}$ are differentiable on I, for $i = 1, 2, \ldots, n$. Then,

$$h(t) = \sigma(t) \cdot \gamma(t) = \sum_{i=1}^{n} x_i(t) y_i(t), \quad \text{for all } t \in I. \tag{5}$$

It follows from (5) and the product rule in Calculus I that h is differentiable and

$$\begin{aligned} h'(t) &= \sum_{i=1}^{n} [x_i'(t) y_i(t) + x_i(t) y_i'(t)] \\ &= \sum_{i=1}^{n} x_i'(t) y_i(t) + \sum_{i=1}^{n} x_i(t) y_i'(t) \\ &= \sigma'(t) \cdot \gamma(t) + \sigma(t) \cdot \gamma'(t), \end{aligned} \tag{6}$$

for all $t \in I$. This establishes (4). □

11. Let I denote an open interval, and $\sigma \colon I \to \mathbb{R}^n$ be a differentiable path satisfying $\|\sigma(t)\| = c$, a constant, for all $t \in I$. Show that, at any $t \in I$, $\sigma(t)$ is orthogonal to a tangent vector to the path at that t.

***Solution**:* Put $h(t) = \|\sigma(t)\|^2 = \sigma(t) \cdot \sigma(t)$ for all $t \in I$. Thus, if $\|\sigma(t)\| = c$ for all $t \in I$, it follows that

$$h(t) = c^2, \quad \text{for all } t \in I. \tag{7}$$

Differentiating on both sides of (7) and using the result of Problem 10in (4) we obtain from (7) that

$$2\sigma'(t) \cdot \sigma(t) = 0. \quad \text{for all } t \in I$$

from which we get that

$$\sigma'(t) \cdot \sigma(t) = 0. \quad \text{for all } t \in I;$$

in other words, $\sigma(t)$ is orthogonal to a tangent vector to the path, $\sigma'(t)$, at any $t \in I$. □

12. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ be given by $\sigma(t) = (t^{1/3}, t)$ for all $t \in \mathbb{R}$. Show that σ is not differentiable at 0.

    ***Solution**:* For $h \neq 0$, compute

    $$\frac{1}{h}[\sigma(h) - \sigma(0)] \;=\; (h^{-2/3}, 1),$$

    so that

    $$\left\| \frac{1}{h}[\sigma(h) - \sigma(0)] \right\|^2 \;=\; 1 + \frac{1}{|h|^{4/3}}\ ,$$

    and therefore

    $$\lim_{h\to 0} \left\| \frac{1}{h}[\sigma(h) - \sigma(0)] \right\| = +\infty,$$

    which shows that $\lim_{h\to 0} \dfrac{1}{h}[\sigma(h) - \sigma(0)]$ does not exist; hence, σ is not differentiable at 0. $\square$