Review Problems for Exam 2

1. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \dfrac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in \mathbb{R}^n$. What is the gradient of f at u for all $x \in \mathbb{R}^n$?

2. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \|v\|$ for all $v \in \mathbb{R}^n$.

   (a) Show that f is differentiable not differentiable at the origin.

   (b) Let $U = \{v \in \mathbb{R}^n \mid v \neq 0\}$. Show that f is differentiable on the set U and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in U$. What is the gradient of f at u for all $x \in U$?

3. Let U denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at every $x \in U$. Fix x and y in U, and define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = f(x + t(y - x)) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

   (a) Explain why the function g is well defined.

   (b) Show that g is differentiable on $(0,1)$ and that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for} \quad 0 < t < 1.$$

   (c) Use the Mean Value Theorem for derivatives to show that there exists a point z is the line segment connecting x to y such that

$$f(y) - f(x) = D_{\widehat{u}} f(z)\|y - x\|,$$

   where $\widehat{u}$ is the unit vector in the direction of the vector $y - x$; that is, $\widehat{u} = \dfrac{1}{\|y - x\|}(y - x)$.

   (d) Prove that if U is an open and convex subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is differentiable on U with $\nabla f(v) = \mathbf{0}$ for all $v \in U$, then f must be a constant function.

4. Let U denote the set of all points in $\mathbb{R}^3$ excluding the origin, $(0,0,0)$. Define the scalar field $f\colon U \to \mathbb{R}$ by $f(x,y,x) = \dfrac{1}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$ for all $(x,y,z) \in U$.

   Show that f is differentiable in U. Compute ∇f and $\text{div}\nabla f$.

5. Compute the arc length along the portion of the cycloid given by the parametric equations
$$x = t - \sin t \quad \text{and} \quad y = 1 - \cos t, \quad \text{for} \quad t \in \mathbb{R},$$
from the point $(0, 0)$ to the point $(2\pi, 0)$.

6. Let C denote the boundary of the oriented triangle, $T = [(0, 0)(1, 0)(1, 2)]$, in $\mathbb{R}^2$. Evaluate the line integral $\displaystyle\int_C \frac{x^2}{2}\ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x$.

7. Let $F(x, y) = 2x\ \widehat{i} - y\ \widehat{j}$ and R be the square in the xy–plane with vertices $(0, 0)$, $(2, -1)$, $(3, 1)$ and $(1, 2)$. Evaluate $\displaystyle\oint_{\partial R} F \cdot n\ \ \mathrm{d}s$.

8. Evaluate the line integral $\displaystyle\int_{\partial R} (x^4 + y)\ \ \mathrm{d}x + (2x - y^4)\ \ \mathrm{d}y$, where R is the rectangular region
$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3,\ -2 \leqslant y \leqslant 1\},$$
and ∂R is traversed in the counterclockwise sense.

9. Integrate the function given by $f(x, y) = xy^2$ over the region, R, defined by:
$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, 0 \leqslant y \leqslant 4 - x^2\}.$$

10. Let R denote the region in the plane defined by inside of the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$
for $a > 0$ and $b > 0$.

    (a) Evaluate the line integral $\displaystyle\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x$, where ∂R is the ellipse in (1) traversed in the positive sense.

    (b) Use your result from part (a) and the Fundamental Theorem of Calculus to come up with a formula for computing the area of the region enclosed by the ellipse in (1).

11. Evaluate the double integral $\displaystyle\int_R e^{-x^2}\ \mathrm{d}x\ \mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 1.

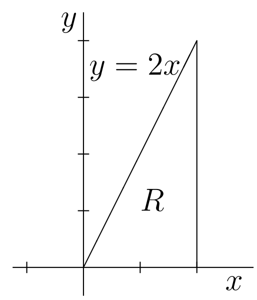

Figure 1: Sketch of Region R in Problem 11

12. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the map from the uv–plane to the xy–plane given by

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v^2 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

and let T be the oriented triangle $[(0,0),(1,0),(1,1)]$ in the uv–plane.

(a) Show that Φ is differentiable and give a formula for its derivative, $D\Phi(u,v)$, at every point $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathbb{R}^2$.

(b) Give the image, R, of the triangle T under the map Φ, and sketch it in the xy–plane.

(c) Evaluate the integral $\displaystyle\iint_R dxdy$, where R is the region in the xy–plane obtained in part (b).

(d) Evaluate the integral $\displaystyle\iint_T |\det[D\Phi(u,v)]|\, dudv$, where $\det[D\Phi(u,v)]$ denotes the determinant of the Jcobian matrix of Φ obtained in part (a). Compare the result obtained here with that obtained in part (c).