Solutions to Review Problems for Exam 2

1. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \dfrac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in \mathbb{R}^n$. What is the gradient of f at u for all $x \in \mathbb{R}^n$?

   Solution: Let u and w be any vector in $\mathbb{R}^n$ and consider

$$
\begin{aligned}
f(u+w) &= \frac{1}{2}\|u+w\|^2 \\
&= \frac{1}{2}(u+w)\cdot(u+w) \\
&= \frac{1}{2}u\cdot u + u\cdot w + \frac{1}{2}w\cdot w \\
&= \frac{1}{2}\|u\|^2 + u\cdot w + \frac{1}{2}\|w\|^2.
\end{aligned}
$$

   Thus,

$$f(u+w) - f(u) - u\cdot w = \frac{1}{2}\|w\|^2.$$

   Consequently,

$$\frac{|f(u+w) - f(u) - u\cdot w|}{\|w\|} = \frac{1}{2}\|w\|,$$

   for $w \in \mathbb{R}^n$ with $\|w\| \neq 0$, from which we get that

$$\lim_{\|w\|\to 0} \frac{|f(u+w) - f(u) - u\cdot w|}{\|w\|} = 0,$$

   and therefore f is differentiable at u with derivative map $Df(u)$ given by

$$Df(u)w = u\cdot w \quad \text{for all } w \in \mathbb{R}^n.$$

   Hence, $\nabla f(u) = u$ for all $u \in \mathbb{R}^n$. □

   Alternate Solution: Write $f(x_1, x_2, \ldots, x_n) = \dfrac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)$ for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then, all the partial derivatives,

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \ldots, x_n) = x_j, \qquad \text{for } j = 1, 2, \ldots, n,$$

are continuous. Thus, f is a C^1 map and is, therefore, differentiable with derivative given by

$$Df(x_1, x_2, \ldots, x_n)h = \nabla f(x_1, x_2, \ldots, x_n) \cdot h, \quad \text{for all } h \in \mathbb{R}^n,$$

where $\nabla f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$ for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. □

2. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \|v\|$ for all $v \in \mathbb{R}^n$.

   (a) Show that f is differentiable not differentiable at the origin.

   Solution: Arguing by contradiction, assume that f is differentiable at the origin. Then, there exists a linear transformation, $T\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$f(w) = T(w) + E_o(w), \tag{1}$$

   for $\|w\|$ small, where

$$\lim_{\|w\|\to 0} \frac{\|E_o(w)\|}{\|w\|} = 0. \tag{2}$$

   Take $w = te_j$, where e_j is one of the standard basis vectors. It then follows from (1) that

$$|t| = tT(e_j) + E_o(te_j),$$

   for $t \in \mathbb{R}$ with $|t|$ sufficiently small. Thus, if $t \neq 0$ and $|t|$ is sufficiently small,

$$\frac{|t|}{t} = T(e_j) + \frac{1}{t}E_o(te_j).$$

   Observe that, by (2),

$$\lim_{t\to 0} \frac{1}{t}E_o(te_j) = 0.$$

   Hence,

$$\lim_{t\to 0} \frac{|t|}{t} = T(e_j),$$

   which is impossible since $\displaystyle\lim_{t\to 0} \frac{|t|}{t}$ does not exist. Consequently, $f(v) = \|v\|$ is not differentiable at the origin. □

(b) Let $U = \{v \in \mathbb{R}^n \mid v \neq 0\}$. Show that f is differentiable on the set U and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in U$. What is the gradient of f at u for all $x \in U$?

Solution: For $v = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, write

$$f(v) = f(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and observe that if $(x_1, x_2, \ldots, x_n) \in U$, then $x_1^2 + x_2^2 + \cdots + x_n^2 \neq 0$ so that the partial derivatives

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \ldots, x_n) = \frac{x_j}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}, \quad j = 1, 2, \ldots, n,$$

exist in U and are continuous there. Therefore, f is a C^1 map in U and it is therefore differentiable in U.
The gradient of f in U is then given by

$$\nabla f(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}(x_1, x_2, \ldots, x_n),$$

or

$$\nabla f(u) = \frac{1}{\|u\|}\, u, \quad \text{for all } u \in U.$$

We therefore have that the derivative map of f at $u \in U$ is given by

$$Df(u)h = \frac{1}{\|u\|}\, u \cdot h, \quad \text{for all } h \in \mathbb{R}^n.$$

□

3. Let U denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at every $x \in U$. Fix x and y in U, and define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = f(x + t(y - x)) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

(a) Explain why the function g is well defined.

Answer: Since U is convex, for any $x, y \in U$, $x + t(y - x) \in U$ for all $t \in [0, 1]$. Thus, $f(x + t(y - x))$ is defined for all $t \in [0, 1]$, because f is defined on U. □

(b) Show that g is differentiable on $(0, 1)$ and that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for} \quad 0 < t < 1.$$

Solution: It follows from the Chain Rule that the composition $g = f \circ \sigma \colon [0,1] \to \mathbb{R}$, where $\sigma \colon [0,1] \to \mathbb{R}^n$ is the path given by

$$\sigma(t) = x + t(y - x), \quad \text{for all } t \in [0,1],$$

is differentiable and

$$g'(t) = \nabla f(\sigma(t)) \cdot \sigma'(t), \quad \text{for all } t \in (0,1),$$

where

$$\sigma(t) = y - x, \quad \text{for all } t.$$

Consequently, we get that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for } \ 0 < t < 1.$$

□

(c) Use the Mean Value Theorem for derivatives to show that there exists a point z is the line segment connecting x to y such that

$$f(y) - f(x) = D_{\widehat{u}} f(z) \|y - x\|, \tag{3}$$

where $\widehat{u}$ is the unit vector in the direction of the vector $y - x$; that is, $\widehat{u} = \dfrac{1}{\|y - x\|}(y - x)$.

Solution: The mean value theorem implies that there exists $\tau \in (0,1)$ such that

$$g(1) - g(0) = g'(\tau)(1 - 0),$$

so that

$$f(y) - f(x) = \nabla f(x + \tau(y - x)) \cdot (y - x). \tag{4}$$

Put $z = x + \tau(y - x)$ and $\widehat{u} = \dfrac{1}{\|y - x\|}(y - x)$. We can then write (4) as

$$\begin{aligned} f(y) - f(x) &= \left(\nabla f(z) \cdot \frac{1}{\|y - x\|}(y - x) \right) \|y - x\| \\ &= (\nabla f(z) \cdot \widehat{u}) \, \|y - x\|, \end{aligned}$$

which yields (3). □

(d) Prove that if U is an open and convex subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is differentiable on U with $\nabla f(v) = \mathbf{0}$ for all $v \in U$, then f must be a constant function.

Solution: Fix $x_o \in U$. Then, for any $x \in U$, the formula in (3) yields

$$f(x) - f(x_o) = D_{\widehat{u}} f(z) \|x - x_o\|, \tag{5}$$

where $D_{\widehat{u}} f(z) = \nabla f(z) \cdot \widehat{u} = 0$ by the assumption. Hence, it follows from (5) that

$$f(x) = f(x_o), \qquad \text{for all } x \in U;$$

in other words, f is constant in U. □

4. Let U denote the set of all points in $\mathbb{R}^3$ excluding the origin, $(0,0,0)$. Define the scalar field $f\colon U \to \mathbb{R}$ by $f(x,y,x) = \dfrac{1}{r}$, where $r = \sqrt{x^2+y^2+z^2}$ for all $(x,y,z) \in U$.

   Show that f is differentiable in U. Compute ∇f and div∇f.

Solution: Write $f(x,y,z) = g(r)$, where $g(r) = \dfrac{1}{r}$, for $r \neq 0$, and $r = \|(x,y,z)\|$ for all $(x,y,z) \in \mathbb{R}^3$. It follows from the result of Problem 2b in this review sheet that r is differentiable for $(x,y,z \in U$, and

$$\nabla r = \frac{1}{r}(x,y,z), \qquad \text{for all } (x,y,z) \in U.$$

Next, note that g is differentiable for $r \neq 0$ and

$$g'(r) = -\frac{1}{r^2}, \qquad \text{for all } r \neq 0.$$

Since f is the composition of f and r, it follows by the Chain Rule that f is differentiable for $(x,y,z) \in U$, and

$$\nabla f(x,y,z) = g'(r)\nabla r = -\frac{1}{r^2} \cdot \frac{1}{r}(x,y,z), \qquad \text{for all } (x,y,z) \in U,$$

or

$$\nabla f(x,y,z) = g'(r)\nabla r = -\frac{1}{r^3}(x,y,z), \qquad \text{for all } (x,y,z) \in U.$$

Next, compute the divergence of ∇f:

$$\text{div}\nabla f(x,y,z) = -\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right), \tag{6}$$

where

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{r^3 - x\cdot 3r^2\dfrac{\partial r}{\partial x}}{r^6}$$

$$= \frac{r^3 - x\cdot 3r^2\dfrac{x}{r}}{r^6},$$

so that

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{r^2-3x^2}{r^5}. \tag{7}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = \frac{r^2-3y^2}{r^5}, \tag{8}$$

and

$$\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) = \frac{r^2-3z^2}{r^5}. \tag{9}$$

Substituting (7)–(9) into (6) then yields

$$\text{div}\nabla f(x,y,z) = -\frac{3r^2 - 3(x^2+y^2+z^2)}{r^5} = 0.$$

□

5. Compute the arc length along the portion of the cycloid given by the parametric equations

$$x = t - \sin t \quad \text{and} \quad y = 1-\cos t, \quad \text{for} \quad t\in\mathbb{R},$$

from the point $(0,0)$ to the point $(2\pi,0)$.

***Solution**:* Put

$$\sigma(t) = (t-\sin t,\ 1-\cos t), \qquad \text{for all } t\in[0,2\pi].$$

Then,

$$\sigma'(t) = (1-\cos t,\ \sin t), \qquad \text{for all } t\in(0,2\pi);$$

so that

$$\|\sigma'(t)\| = \sqrt{(1-\cos t)^2+\sin^2 t}$$

$$= \sqrt{1-2\cos t+\cos^2 t+\sin^2 t} \tag{10}$$

$$= \sqrt{2-2\cos t}.$$

Next, use the trigonometric identity

$$2\sin^2\left(\frac{t}{2}\right) = 1 - \cos t,$$

to obtain from the calculations in (10) that

$$\begin{aligned} \|\sigma'(t)\| &= \sqrt{4\sin^2\left(\frac{t}{2}\right)} \\ &= 2\left|\sin\left(\frac{t}{2}\right)\right|, \end{aligned} \tag{11}$$

for $t \in (0, 2\pi)$. Now, since since $0 \leqslant \dfrac{t}{2} \leqslant \pi$ for $0 \leqslant t \leqslant 2\pi$, it follows that

$$\sin\left(\frac{t}{2}\right) \geqslant 0, \qquad \text{for } t \in [0, 2\pi].$$

We then obtain from (11) that

$$\|\sigma'(t)\| = 2\sin\left(\frac{t}{2}\right), \qquad \text{for all } t \in [0, 2\pi].$$

Consequently, the arc length along the portion of the cycloid parametrized by $\sigma(t)$ for $0 \leqslant t \leqslant 2\pi$ is

$$\begin{aligned} \int_0^{2\pi} \|\sigma'(t)\| \, dt &= \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) \, dt \\ &= \left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} \\ &= 8. \end{aligned}$$

□

6. Let C denote the boundary of the oriented triangle, $T = [(0,0)(1,0)(1,2)]$, in $\mathbb{R}^2$. Evaluate the line integral $\displaystyle\int_C \frac{x^2}{2}\, dy - \frac{y^2}{2}\, dx$.

***Solution**:* First observe that $\displaystyle\int_C \frac{x^2}{2}\, dy - \frac{y^2}{2}\, dx$ is the flux of the vector field

$$F(x,y) = \left(\frac{x^2}{2}, \frac{y^2}{2}\right)$$

across the boundary of T. Thus, applying the divergence form of Fundamental Theorem of Calculus,

$$\int_{\partial T} F \cdot \widehat{n}\, ds = \iint_T \text{div} F\, dxdy,$$

we obtain that

$$\begin{aligned}\int_C \frac{x^2}{2}\ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x &= \iint_T (x+y)\, dxdy \\ &= \int_0^1 \int_0^{2x} (x+y)\, dydx \\ &= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{2x} dx \\ &= \int_0^1 4x^2 dx,\end{aligned}$$

so that

$$\int_C \frac{x^2}{2}\ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x = \frac{4}{3}.$$

□

7. Let $F(x,y) = 2x\ \widehat{\imath} - y\ \widehat{\jmath}$ and R be the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\oint_{\partial R} F \cdot n\ \ \mathrm{d}s$.

***Solution**:* Apply the divergence form of the Fundamental Theorem of Calculus to get

$$\oint_{\partial R} F \cdot \widehat{n}\, ds = \iint_R \text{div} F\, dxdy,$$

where

$$\text{div} F(x,y) = 2 - 1 = 1,$$

so that

$$\begin{aligned}\oint_{\partial R} F \cdot \widehat{n}\, ds &= \iint_R dxdy \\ &= \text{area}(R).\end{aligned}$$

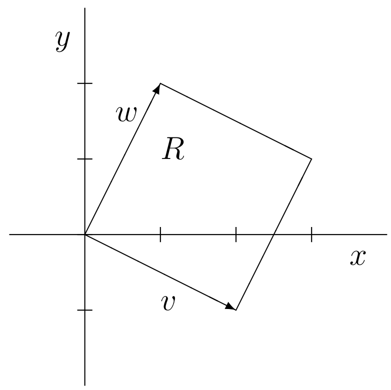

Figure 1: Sketch of Region R in Problem 7

To find the area of the region R, shown in Figure 1, observe that R is a parallelogram determined by the vectors $v = 2\,\widehat{i} - \widehat{j}$ and $w = \widehat{i} + 2\,\widehat{j}$. Thus,

$$\text{area}(R) = \|v \times w\| = 5.$$

It the follows that

$$\oint_{\partial R} F \cdot n \ \ \mathrm{d}s = \iint_R \mathrm{d}x \ \mathrm{d}y = 5.$$

□

8. Evaluate the line integral $\displaystyle\int_{\partial R} (x^4 + y) \ \ \mathrm{d}x + (2x - y^4) \ \ \mathrm{d}y$, where R is the rectangular region

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3, \ -2 \leqslant y \leqslant 1\},$$

and ∂R is traversed in the counterclockwise sense.

Solution: Apply the Green's Theorem form of Fundamental Theorem of Calculus to get

$$\begin{aligned}
\int_{\partial R} (x^4 + y)\ dx + (2x - y^4)\ dy &= \iint_R \left(\frac{\partial}{\partial x}(2x - y^4) - \frac{\partial}{\partial y}(x^4 + y \right)\ dxdy \\
&= \iint_R (2 - 1)\ dxdy \\
&= \iint_R dxdy \\
&= \text{area}(R) \\
&= 12.
\end{aligned}$$

□

9. Integrate the function given by $f(x, y) = xy^2$ over the region, R, defined by:

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, 0 \leqslant y \leqslant 4 - x^2\}.$$

***Solution**:* The region, R, is sketched in Figure 2. We evaluate the

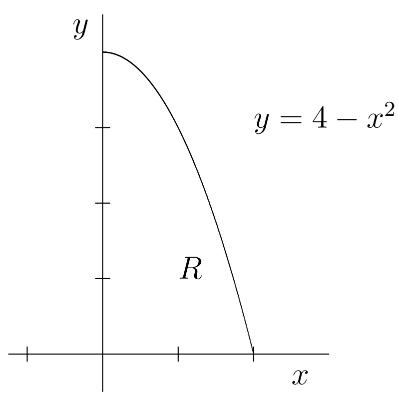

Figure 2: Sketch of Region R in Problem 9

double integral, $\iint_R xy^2 \, \mathrm{d}x \, \mathrm{d}y$, as an iterated integral

$$\begin{aligned}
\iint_R xy^2 \, \mathrm{d}x \, \mathrm{d}y &= \int_0^2 \int_0^{4-x^2} xy^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^2 \int_0^{4-x^2} xy^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^2 \frac{xy^3}{3}\Big|_0^{4-x^2} \, \mathrm{d}x \\
&= \frac{1}{3}\int_0^2 x(4-x^2)^3 \, \mathrm{d}x.
\end{aligned}$$

To evaluate the last integral, make the change of variables: $u = 4-x^2$. We then have that $\mathrm{d}u = -2x \, \mathrm{d}x$ and

$$\begin{aligned}
\iint_R xy^2 \, \mathrm{d}x \, \mathrm{d}y &= \int_0^2 \int_0^{4-x^2} xy^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= -\frac{1}{6}\int_4^0 u^3 \, \mathrm{d}u \\
&= \frac{1}{6}\int_0^4 u^3 \, \mathrm{d}u.
\end{aligned}$$

Thus,

$$\iint_R xy^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{4^4}{24} = \frac{32}{3}.$$

□

10. Let R denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{12}$$

for $a > 0$ and $b > 0$.

(a) Evaluate the line integral $\oint_{\partial R} x \, \mathrm{d}y - y \, \mathrm{d}x$, where ∂R is the ellipse in (12) traversed in the positive sense.

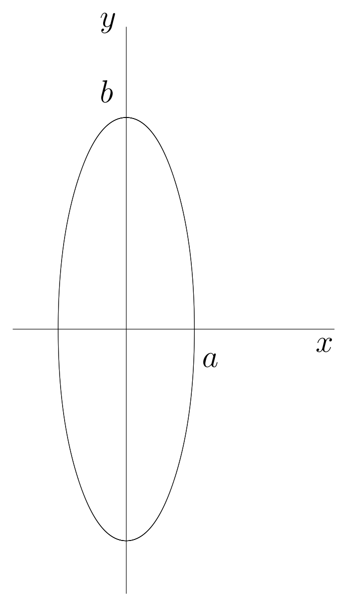

Figure 3: Sketch of ellipse

Solution: A sketch of the ellipse is shown in Figure 3 for the case $a < b$.
A parametrization of the ellipse is given by

$$x = a\cos t, \quad y = b\sin t, \quad \text{for } \ 0 \leqslant t \leqslant 2\pi.$$

We then have that $\mathrm{d}x = -a\sin t\,\mathrm{d}t$ and $\mathrm{d}y = b\cos t\,\mathrm{d}t$. Therefore

$$\begin{aligned}
\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x &= \int_0^{2\pi} [a\cos t \cdot b\cos t - b\sin t\cdot(-a\cos t)]\ \mathrm{d}t \\
&= \int_0^{2\pi} [ab\cos^2 t + ab\sin^2 t]\ \mathrm{d}t \\
&= ab\int_0^{2\pi} (\cos^2 t + ab\sin^2 t)\ \mathrm{d}t \\
&= ab\int_0^{2\pi}\ \mathrm{d}t \\
&= 2\pi ab.
\end{aligned}$$

□

(b) Use your result from part (a) and the Fundamental Theorem of Calculus to come up with a formula for computing the area of the region enclosed by the ellipse in (12).

Solution: Let $F(x, y) = x\,\widehat{\imath} + y\,\widehat{\jmath}$. Then,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = \oint_{\partial R} F \cdot n\ \ \mathrm{d}s.$$

Thus, by Green's Theorem in divergence form,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = \iint_R \mathrm{div}F\ \ \mathrm{d}x\ \mathrm{d}y,$$

where

$$\mathrm{div}F(x, y) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2.$$

Consequently,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = 2\iint_R\ \mathrm{d}x\ \mathrm{d}y = 2\ \mathrm{area}(R).$$

It then follows that

$$\mathrm{area}(R) = \frac{1}{2}\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x.$$

Thus,

$$\mathrm{area}(R) = \pi ab,$$

by the result in part (a). □

11. Evaluate the double integral $\int_R e^{-x^2}\ \mathrm{d}x\ \mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 4.

Solution: Compute

$$\begin{aligned}
\iint_R e^{-x^2}\ \ \mathrm{d}x\ \mathrm{d}y &= \int_0^2 \int_0^{2x} e^{-x^2}\ \ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^2 2xe^{-x^2}\ \ \mathrm{d}x \\
&= \left[-e^{-x^2}\right]_0^2 \\
&= 1 - e^{-4}.
\end{aligned}$$

□

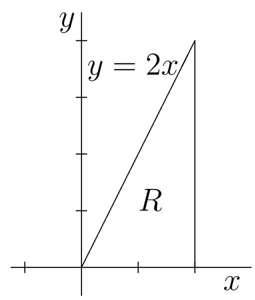

Figure 4: Sketch of Region R in Problem 11

12. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the map from the uv–plane to the xy–plane given by

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v^2 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

and let T be the oriented triangle $[(0,0),(1,0),(1,1)]$ in the uv–plane.

(a) Show that Φ is differentiable and give a formula for its derivative, $D\Phi(u,v)$, at every point $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathbb{R}^2$.

Solution: Write

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

where $f(u,v) = 2u$ and $g(u,v) = v^2$ for all $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$. Observe that the partial derivatives of f and g exist and are given by

$$\frac{\partial f}{\partial u}(u,v) = 2, \qquad \frac{\partial f}{\partial v}(u,v) = 0$$

$$\frac{\partial g}{\partial u}(u,v) = 0, \qquad \frac{\partial g}{\partial v}(u,v) = 2v.$$

Note that the partial derivatives of f and g are continuous. Therefore, Φ is a C^1 map. Hence, Φ is differentiable on $\mathbb{R}^2$ and its derivative map at (u,v), for any $(u,v) \in \mathbb{R}^2$ is given by multiplication by the Jacobian matrix

$$D\Phi(u,v) = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix};$$

that is,

$$D\Phi(u,v)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2h \\ 2vk \end{pmatrix}$$

for all $\begin{pmatrix} h \\ k \end{pmatrix} \in \mathbb{R}^2$. □

(b) Give the image, R, of the triangle T under the map Φ, and sketch it in the xy–plane.

Solution: The image of T under Φ is the set

$$\begin{aligned} \Phi(T) &= \{(x,y) \in \mathbb{R}^2 \mid x = 2u, y = v^2, \text{ for some } (u,v) \in T\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant x^2/4\}. \end{aligned}$$

A sketch of $R = \Phi(T)$ is shown in Figure 5. □

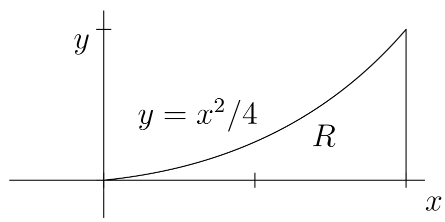

Figure 5: Sketch of Region $\Phi(T)$

(c) Evaluate the integral $\displaystyle\iint_R dxdy$, where R is the region in the xy–plane obtained in part (b).

Solution: Compute by means of iterated integrals

$$\begin{aligned} \iint_R dxdy &= \int_0^2 \int_0^{x^2/4} dy\, dx \\ &= \int_0^2 \frac{x^2}{4}\, dx \\ &= \left[\frac{x^3}{12}\right]_0^2 \\ &= \frac{2}{3}. \end{aligned}$$

□

(d) Evaluate the integral $\iint_T |\det[D\Phi(u,v)]|\ dudv$, where $\det[D\Phi(u,v)]$ denotes the determinant of the Jacobian matrix of Φ obtained in part (a). Compare the result obtained here with that obtained in part (c).

Solution: Compute $\det[D\Phi(u,v)]$ to get

$$\det[D\Phi(u,v)] = 4v.$$

so that

$$\iint_T |\det[D\Phi(u,v)]| dudv = \iint_T 4|v|\ dudv,$$

where the region T, in the uv–plane is sketched in Figure 6. Observe that, in that region, $v \geqslant 0$, so that

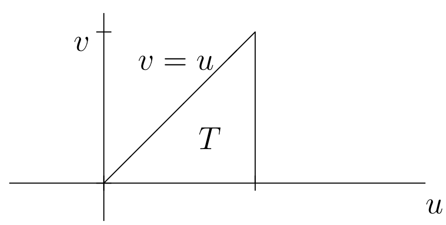

Figure 6: Sketch of Region T

$$\iint_T |\det[D\Phi(u,v)]| dudv = \iint_T 4v\ dudv,$$

Compute by means of iterated integrals

$$\begin{aligned}\iint_T |\det[D\Phi(u,v)]| dudv &= \int_0^1 \int_0^u 4v\ dvdu \\ &= \int_0^1 2u^2\ du \\ &= \frac{2}{3},\end{aligned}$$

which is the same result as that obtained in part (c). □