Exam 1

Wednesday, October 12, 2011

Name: ________________________________

Show all significant work and justify all your answers. This is a closed book exam. Use your own paper and/or the paper provided by the instructor. You have 50 minutes to work on the following 3 problems. Relax.

1. When people smoke, carbon monoxide is released into the air. Suppose that in a room of volume 60 m^3, air containing 5% carbon monoxide is introduced at a rate of 0.002 m^3/min. (This means that 5% of the volume of incoming air is carbon monoxide). The carbon monoxide mixes immediately with the air and the mixture leaves the room at the same rate as it enters.

   (a) Let $Q = Q(t)$ denote the volume (in cubic meters) of carbon monoxide in the room at any time t in minutes. Use a conservation principle to derive a differential equation for Q.

   (b) Give the equilibrium solution, $\overline{Q}$, to the differential equation in part (a).

   (c) Solve the differential equation in part (a) under the assumption that the there is no carbon monoxide in the room initially, and sketch the solution.

   (d) Based on your solution to part (c), give the concentration, $c(t)$, of carbon monoxide in the room (in percent volume) at any time t in minutes. What happens to the value of $c(t)$ in the long run?

   (e) Medical texts warn that exposure to air containing 0.1% carbon monoxide for some time can lead to a coma. How many hours does it take for the concentration of carbon monoxide found in part (d) to reach this level?

2. Suppose that $y = y(t)$ is a solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} &= e^{-t^2}, \quad t \in \mathbf{R}, \\ y(0) &= 0. \end{cases} \tag{1}$$

   (a) Find y' and y''.

   (b) Determine the values of t for which $y(t)$ increases or decreases, and the values of t for which the graph of $y = y(t)$ is concave up or concave down. Sketch the graph of $y = y(t)$ given that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$.

3. Assume that the relative growth rate of a certain animal population is governed by the equation

$$\frac{1}{N}\frac{dN}{dt} = k_o\, e^{-t}, \tag{2}$$

where $N = N(t)$ is the number of individuals in the population t units of time from the time we start observing the population, and k_o is a positive constant.

   (a) Give an interpretation for this model and explain how it differs from the Malthus model for population growth.

   (b) Use separation of variables to find a solution to (2) subject to the initial condition $N(0) = N_o$.

   (c) What does the model predict about the number of individuals in the population in the long run.

   (d) **(Bonus)** Given that the population doubles after one unit of time, find k_o and compute

$$\lim_{t\to\infty} N(t).$$