Solutions Review Problems for Exam #2

1. Suppose that the growth of a population of size $N = N(t)$ follows the differential equation model
$$\frac{dN}{dt} = aN - b, \tag{1}$$
where a and b are positive parameters.

   (a) Give an interpretation for the model in (1).

   ***Solution**:* Equation (1) models a population that undergoes Malthusian growth with a constant *per–capita* growth rate, a, and which is being harvested at a constant rate b. □

   (b) Describe all possible behaviors predicted by the model in (1).

   ***Solution**:* The general solution to equation (1) is
$$N(t) = \frac{b}{a} + ce^{at}, \quad \text{for all } t \in \mathbb{R}. \tag{2}$$
   Thus, since $a > 0$, it follows from (2) that solutions to (1) tend away from the equilibrium value $\overline{N} = \dfrac{b}{a}$. Figure 1 shows three typical solutions. Examination of the sketches in Figure 1 shows that, if the initial population

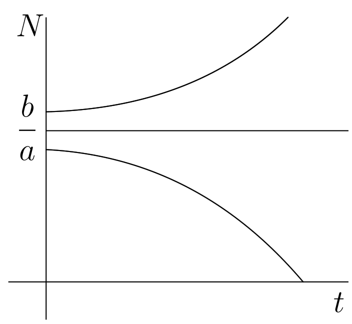

Figure 1: Sketch of possible solutions to (1)

   size, $N_o = N(0)$, is larger than the equilibrium value $\overline{N} = \dfrac{b}{a}$, then the population will experience unlimited exponential growth. On the other hand, if $N_o < \overline{N}$, then the population will cease to exist in finite time. □

2. Find the equilibrium points of the differential equation

$$\frac{dy}{dt} = y^2 - 36, \tag{3}$$

and determine their stability properties.

Solution: Set $f(y) = y^2 - 36$ and write $f(y) = (y+6)(y-6)$; so that, the differential equation in (3) has two equilibrium solutions:

$$\overline{y}_1 = -6 \quad \text{and} \quad \overline{y}_2 = 6.$$

In order to determine the stability of $\overline{y}_1$ and $\overline{y}_2$, we first compute $f'(y) = 2y$. Since $f'(-6) = -12 < 0$, $\overline{y}_1$ is asymptotically stable by the principle of linearized stability; similarly, since $f'(6) = 12 > 0$, $\overline{y}_2$ is unstable by the principle of linearized stability. □

3. We have seen that the (continuous) logistic model $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$, where r and K are positive parameters, has an equilibrium point at $\overline{N} = K$.

   (a) Let $f(N) = rN\left(1 - \dfrac{N}{K}\right)$ and give the linear approximation to $f(N)$ for N close to K.

   Solution: The linear approximation to f at $\overline{N} = K$ is

$$L(N; \overline{N}) = f(K) + f'(K)(N - K) = -r(N - K).$$

□

   (b) Let $u = N - K$ and consider the linear differential equation

$$\frac{du}{dt} = f'(K)u.$$

   This is called the *linearization* of the equation

$$\frac{dN}{dt} = f(N) \tag{4}$$

   around the equilibrium point $\overline{N} = K$.

   Use separation of variables to solve this equation. What happens to $|u(t)|$ as $t \to \infty$, where u is any solution to the linearized equation?

   Solution: The linearization of (4) is

$$\frac{du}{dt} = -ru. \tag{5}$$

Separation of variables leads to the general solution of (5),

$$u(t) = c\, e^{-rt}, \quad \text{for all } t \in \mathbb{R}. \tag{6}$$

Thus, if u is a solution to the linearized equation in (6), then

$$|u(t)| = |c|\, e^{-rt} \to 0, \quad \text{as } t \to \infty, \tag{7}$$

since $r > 0$. □

(c) Use your result in the previous part to give an explanation as to why any solution to the logistic equation that begins very close to K can be approximation by $K+u(t)$, where u is a solution to the linearized equation.

***Solution**:* Let $N = N(t)$ denote a solution to the differential equation in (4), and suppose that $N(0) = N_o$ is very close to K. Put $u = N - K$; then,

$$\begin{aligned} \frac{du}{dt} &= \frac{dN}{dt} \\ &= f(N) \\ &= -r(N-K) + E(N;K), \end{aligned}$$

where $E(N;K)$ denotes the error in the linear approximation. We then have that

$$\frac{du}{dt} = -ru + E(N;K), \tag{8}$$

where

$$\lim_{N\to K} \frac{E(N;K)}{N-K} = 0. \tag{9}$$

It follows from (9) and (8) that, when N_o is very close to K, then the solution, $u = N-K$, to (8) with $N(0) = N_o$ is very close to the solution to the linearized equation (5). Thus, $N(t) - K$ can be approximated by $u(t)$, where u solves the linearized equation in (5) subject to $u(0) = N_o - K$; that is,

$$N(t) - K \approx u(t),$$

or

$$N(t) \approx K + u(t), \tag{10}$$

where u is a solution to the linearized equation (5). □

(d) Suppose that $N = N(t)$ is a solution to the logistic equation that starts at N_o, where N_o is very close to K. Find an estimate of the time it takes for the distance $|N(t) - K|$ to decrease by a factor of e. This time is called the *recovery time.*

***Solution**:* It follows from (10) and (6) that

$$N(t) - K \approx (N_o - K)e^{-rt}, \text{ for all } t > 0.$$

so that

$$|N(t) - K| \approx |N_o - K|e^{-rt}, \text{ for all } t > 0. \tag{11}$$

To find the time, t, when

$$\frac{|N(t) - K|}{|N_o - K|} = \frac{1}{e},$$

we use (11) to obtain the equation

$$e^{-rt} = \frac{1}{e},$$

which can be solved for t to obtain

$$t = \frac{1}{r},$$

the recovery time. □

4. Consider the first–order ordinary differential equation

$$\frac{dy}{dt} = y^2 - 2y + 1. \tag{12}$$

(a) Determine equilibrium points and determine the nature of the stability of the equilibrium solutions by means of the principle of linearized stability

***Solution**:* Put $f(y) = y^2 - 2y + 1$ and write $f(y) = (y - 1)^2$; so that, the differential equation in (12) has one equilibrium solution; namely,

$$\overline{y} = 1.$$

Since $f'(y) = 2(y-1)$, $f'(1) = 0$, so that the principle of linearized stability does not apply in this case. □

(b) Use separation of variables to find the general solution to the equation.

***Solution**:* Use separation of variables to solve the equation

$$\frac{dy}{dt} = (y-1)^2.$$

We obtain

$$\int \frac{1}{(y-1)^2}\,dy = \int dt,$$

which yields

$$-\frac{1}{y-1} = t + c_1, \tag{13}$$

for some arbitrary constant c_1. Multiply on both sides of the equation in (13) by -1 and solve for y to obtain

$$y(t) = 1 + \frac{1}{c-t}, \tag{14}$$

for some arbitrary constant c. □

(c) Use your result from the previous part to determine the nature of the stability of the equilibrium points.

***Solution**:* Let y_o be such that $y_o > 1$, and assume that a solution $y = y(t)$ to the differential equation in (12) satisfies $y(0) = y_o$. We then obtain from (14) that

$$c = \frac{1}{y_o - 1}. \tag{15}$$

Substituting the value for c in (15) into (14) yields the solution

$$y(t) = 1 + \frac{y_o - 1}{1 - (y_o - 1)t} \tag{16}$$

to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} &= y^2 - 2y + 1; \\ y(0) &= y_o, \end{cases} \tag{17}$$

which ceases to exist at $t = \dfrac{1}{y_o - 1}$. Therefore, for $y_o > 1$, the solution the the IVP in (17) does not exist for all $t > 0$. Hence, $\overline{y} = 1$ is unstable. □

(d) Find a solution to the IVP $\begin{cases} \dfrac{dy}{dt} & = & y^2 - 2y + 1; \\ y(0) & = & 2, \end{cases}$ and determine its maximal interval of existence.

Solution: Using the formula in (16) derived in the previous part we see that the solution to the IVP in (17) for $y_o = 2$ is given by

$$y(t) = 1 + \frac{1}{1-t}, \qquad \text{for } t < 1.$$

Thus, the maximal interval of existence is $(-\infty, 1)$. □

5. Let $F(t) = \displaystyle\int_0^t \tau^2 e^{-\tau} \mathrm{d}\tau$ for all $t \in \mathbb{R}$.

(a) Use integration by parts to evaluate $F(t)$.

Solution: Set

$$\begin{array}{rcl} u = \tau^2 & \text{and} & dv = e^{-\tau}\, d\tau \\ \text{then, } du = 2\tau\, d\tau & \text{and} & v = -e^{-\tau}, \end{array}$$

so that

$$\int \tau^2 e^{-\tau}\, d\tau = -\tau^2 e^{-\tau} + \int 2\tau e^{-\tau}\, d\tau. \tag{18}$$

We integrate by parts the integral on the right–hand side of (18) by setting

$$\begin{array}{rcl} u = 2\tau & \text{and} & dv = e^{-\tau}\, d\tau \\ \text{then, } du = 2\, d\tau & \text{and} & v = -e^{-\tau}, \end{array}$$

so that

$$\int \tau^2 e^{-\tau}\, d\tau = -\tau^2 e^{-\tau} - 2\tau e^{-\tau} + 2\int e^{-\tau}\, d\tau,$$

from which we get that

$$\int \tau^2 e^{-\tau}\, d\tau = -\tau^2 e^{-\tau} - 2\tau e^{-\tau} - 2e^{-\tau} + c, \tag{19}$$

for arbitrary c. Using the result in (19) we obtain that

$$\begin{aligned} F(t) &= \int_0^t \tau^2 e^{-\tau}\, d\tau \\ &= \left[-\tau^2 e^{-\tau} - 2\tau e^{-\tau} - 2e^{-\tau}\right]_0^t, \end{aligned}$$

which yields the formula

$$F(t) = 2 - t^2e^{-t} - 2be^{-t} - 2e^{-t} \tag{20}$$

for computing $F(t)$, for $t > 0$. □

(b) Sketch the graph of $y = F(t)$.

***Solution**:* It follows from the definition of $F(t)$ and the Fundamental Theorem of Calculus that

$$F'(t) = t^2e^{-t}, \quad \text{for all } t \in \mathbb{R},$$

so that $F'(t) > 0$ for all $t \neq 0$. Thus, $F(t)$ is increasing as t in creases. Next, compute the second derivative of F to obtain

$$F''(t) = t(2-t)e^{-t}, \quad \text{for all } t \in \mathbb{R}. \tag{21}$$

We see from the expression for $F''(t)$ in (21) that the sign of $F''(t)$ is determined by the signs of the factors, t and $2 - t$. The signs of these factors are displayed in Table 1. The concavity of the graph of $y = F(t)$ is

t:	$-$	$+$	$+$
$2-t$:	$+$	$+$	$-$
	0	2	t
$f''(t)$:	$-$	$+$	$-$
Concavity:	down	up	down

Table 1: Concavity of the graph of $y = F(t)$

also shown in Table 1. From the information in the table, we also conclude that the graph of $y = F(t)$ has inflection points at the points when $t = 0$ and $t = 2$. A sketch of the graph of $y = F(t)$ is shown in Figure 2. □

6. Let $g \colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$

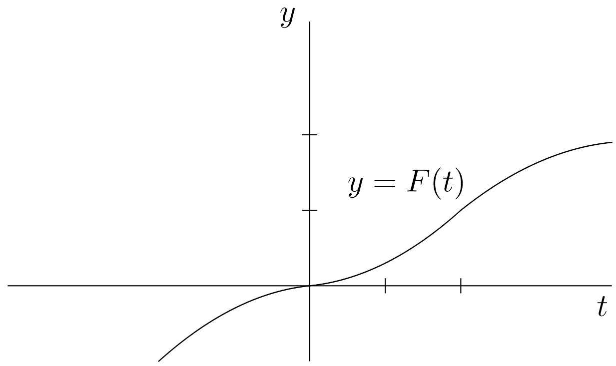

Figure 2: Sketch of graph of $y = F(t)$

(a) Use the first linear approximation to sin around $a = 0$, with the corresponding error term, to compute $\lim_{x\to 0} \dfrac{\sin x}{x}$, and conclude that the function g defined above is continuous.

***Solution**:* Set $f(x) = \sin x$ for all $x \in \mathbb{R}$. Then, the linear approximation to $f(x) = \sin x$ is

$$L(x; 0) = f(0) + f'(0)x,$$

where $f(0) = 0$ and $f'(0) = \cos 0 = 1$. We then have that the linear approximation to $f(x) = \sin x$ at $a = 0$ is

$$L(x; 0) = x. \tag{22}$$

We can then write that

$$\sin x = x + E_f(x; 0), \tag{23}$$

where the error term, $E_f(x; 0)$, in using the linear approximation in (22) to estimate $\sin x$, for x near 0, satisfies

$$\lim_{x\to 0} \frac{E_f(x; 0)}{x} = 0. \tag{24}$$

Next, divide the expression in (23) by x, where $x \neq 0$, to get that

$$\frac{\sin x}{x} = 1 + \frac{E_f(x; 0)}{x}, \qquad \text{for } x \neq 0. \tag{25}$$

It then follows from (24) and (25) that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{26}$$

From (26) and the definition of g we get that

$$\lim_{x\to 0} g(x) = 1 = g(0),$$

which shows that g is continuous at 0. Since, for $x \neq 0$, g is the ratio of two continuous functions, whose denominator is not 0 for $x \neq 0$, it also follows that g is continuous everywhere. □

(b) Use the first order approximation to sin around $a = 0$ to find an approximation for g around $a = 0$. Estimate the error in the approximation.

***Solution**:* It follows from (25) and the definition of g that

$$g(x) = 1 + \frac{E_f(x;0)}{x}, \qquad \text{for } x \neq 0.$$

We therefore have that

$$g(x) = 1 + E_g(x;0), \qquad \text{for } x \neq 0, \tag{27}$$

where the error term, $E_g(x;0)$, in (27) is defined by

$$E_g(x;0) = \frac{E_f(x;0)}{x}, \qquad \text{for } x \neq 0. \tag{28}$$

Thus, the first order approximation to $g(x)$ around 0 is 1, with error term given by (28).

In order to estimate the error term, $E_g(x;0)$, we first estimate $E_f(x;0)$ by

$$|E_f(x;0)| \leqslant \frac{M}{2}|x|^2,$$

where we can take $M = 1$, since $|f''(x)| = |\sin x| \leqslant 1$ for all $x \in \mathbb{R}$. We then have that

$$|E_f(x;0)| \leqslant \frac{1}{2}|x|^2. \tag{29}$$

Using the estimate for $E_f(x;0)$ in (29), we obtain from (28) that

$$|E_g(x;0)| \leqslant \frac{1}{2}|x|, \qquad \text{for all } x \in \mathbb{R}. \tag{30}$$

□

(c) Use the result in (b) above to approximate $\displaystyle\int_0^x \frac{\sin t}{t}\,\mathrm{d}t$. How good is your approximation?

Solution: Note that

$$\int_0^x \frac{\sin t}{t}\, dt = \int_0^x g(t)\, dt. \tag{31}$$

Thus, in order to estimate $\displaystyle\int_0^x \frac{\sin t}{t} \mathrm{d}t$, we can use the estimate for g given in (27).

It follows from (31) and (27) that

$$\int_0^x \frac{\sin t}{t}\, dt = x + \int_0^x E_g(t;0)\, dt, \tag{32}$$

where $E_g(t;0)$ satisfies the estimate in (30); namely

$$|E_g(t;0)| \leqslant \frac{1}{2}|t|, \quad \text{for all } t \in \mathbb{R}. \tag{33}$$

Put

$$E(x;0) = \int_0^x E_g(t;0)\, dt, \quad \text{for all } x \in \mathbb{R}. \tag{34}$$

We then have from (32) that

$$\int_0^x \frac{\sin t}{t}\, dt = x + E(x;0), \text{ for all } x \in \mathbb{R}. \tag{35}$$

Thus, according to (35), we can approximate $\displaystyle\int_0^x \frac{\sin t}{t} \mathrm{d}t$ by x, and the error in this approximation, $E(x,0)$, for $x > 0$, can be estimated from (34) and (33) as follows:

$$\begin{aligned} |E(x;0)| &\leqslant \int_0^x |E_g(t;0)|\, dt \\ &\leqslant \int_0^x \frac{1}{2}|t|\, dt \\ &\leqslant \frac{1}{4}|x|^2. \end{aligned}$$

A similar calculation for $x < 0$ shows that

$$|E(x;0)| \leqslant \frac{1}{4}|x|^2, \quad \text{for all } x \in \mathbb{R}.$$

□

7. Solve the initial value problem

$$\frac{dy}{dt} = y + t^2, \qquad y(0) = 0,$$

and compute $\lim_{t\to\infty} y(t)$.

***Solution**:* Rewrite the equation as

$$\frac{dy}{dt} - y = t^2$$

and multiply by e^{-t} to obtain

$$e^{-t}\frac{dy}{dt} - e^{-t}y = t^2e^{-t},$$

which can be written as

$$\frac{d}{dt}[e^{-t}y] = t^2e^{-t}, \tag{36}$$

by virtue of the product rule. Integrating on both sides of (36) yields

$$e^{-t}y = \int t^2e^{-t}\ dt. \tag{37}$$

In order to evaluate the integral on the right–hand side of (37), we use the result of Problem 5 in this review sheet to get

$$\int te^t\ dt = 2 - (t^2 + 2t + 2)e^{-t} + c, \tag{38}$$

where c is an arbitrary constant. Substituting the result in (38) into the right–hand side of (37) yields

$$e^{-t}y = 2 - (t^2 + 2t + 2)e^{-t} + c \tag{39}$$

Solving for y in (40) we obtain

$$y(t) = 2e^t - t^2 - 2t - 2 + ce^t, \qquad \text{for all } t \in \mathbb{R}. \tag{40}$$

Using the initial condition, $y(0) = 0$, in (40) we have that $c = 0$. Thus,

$$y(t) = 2e^t - t^2 - 2t - 2, \qquad \text{for all } t \in \mathbb{R}. \tag{41}$$

It follows from (41) that $\lim_{t\to\infty} y(t) = +\infty$. □

8. Solve the initial value problem

$$\frac{dy}{dt} = e^t \sin t, \qquad y(0) = 0.$$

***Solution**:* The solution to the initial value problem is

$$y(t) = \int_0^t e^\tau \sin\tau \, d\tau, \quad \text{for all } t \in \mathbb{R}. \tag{42}$$

In order to evaluate the integral on the right–hand side of (42), we use integration by parts. Set

$$\begin{aligned} u = \sin\tau \quad &\text{and} \quad dv = e^\tau \, d\tau \\ \text{then, } du = \cos\tau \, d\tau \quad &\text{and} \quad v = e^\tau, \end{aligned}$$

so that

$$\int e^\tau \sin\tau \, d\tau = e^\tau \sin\tau - \int e^\tau \cos\tau d\tau. \tag{43}$$

Integrate by parts the right-most integral in (43) by setting

$$\begin{aligned} u = \cos\tau \quad &\text{and} \quad dv = e^\tau \, d\tau \\ \text{so that, } du = -\sin\tau \, d\tau \quad &\text{and} \quad v = e^\tau. \end{aligned}$$

We then get from (43) that

$$\int e^\tau \sin\tau \, d\tau = e^\tau \sin\tau - \left[e^\tau \cos\tau + \int e^\tau \sin\tau d\tau \right],$$

or

$$\int e^\tau \sin\tau \, d\tau = e^\tau \sin\tau - e^\tau \cos\tau - \int e^\tau \sin\tau d\tau. \tag{44}$$

Adding $\int e^\tau \sin\tau d\tau$ on both sides of (44) and dividing by 2 then yields the integration formula

$$\int e^\tau \sin\tau \, d\tau = \frac{e^\tau}{2}[\sin\tau - \cos\tau] + c, \tag{45}$$

where we have added an arbitrary constant c. We can now use the integration formula in (45) to evaluate $y(t)$ in (42):

$$\begin{aligned} y(t) &= \left[\frac{e^\tau}{2}[\sin\tau - \cos\tau] \right]_0^t \\ &= \frac{e^t}{2}(\sin t - \cos t) + \frac{1}{2}. \end{aligned}$$

□

9. Consider the first order differential equation

$$\frac{dy}{dt} = y^3 - 4y. \tag{46}$$

   (a) Find all equilibrium solutions of the equation and determine the nature of their stability.

   ***Solution**:* Set $f(y) = y^3 - 4y$ and write

$$f(y) = y(y+2)(y-2).$$

   The the differential equation in (46) has three equilibrium solutions:

$$\overline{y}_1 = -2, \quad \overline{y}_2 = 0 \quad \text{and} \quad \overline{y}_3 = 2.$$

   To determine the stability properties of the equilibrium points, we compute

$$f'(y) = 3y^2 - 4,$$

   and evaluate

$$f'(-2) = 8, \quad f'(0) = -4, \quad \text{and} \quad f'(2) = 8.$$

   Thus, $f'(-2) > 0$, so that $\overline{y}_1 = -2$ is unstable; $f'(0) < 0$, so that $\overline{y}_2 = 0$ is asymptotically stable; and $f'(2) > 0$, so that $\overline{y}_3 = 2$ is unstable, by the principle of linearized stability. □

   (b) Sketch a few of the possible solutions to the equation.

   ***Solution**:* Figure 3 shows a few possible solutions of the differential equation in (46). □

10. The law of mass action states that the rate of a chemical reaction is proportional to the concentrations of the reacting substances.

    Consider a chemical reaction, $A + B \to C$, in which two substances, A and B, react to produce a single substance, C. Assume that the reverse reaction does not have a considerable effect and therefore can be neglected. Let $y = y(t)$ denote the number of kilograms of the reaction product, C, after t minutes. Suppose that the original amount of the reacting substances are 80 kilograms and 60 kilograms. As a consequence of the law of mass action, we obtain that

$$\frac{dy}{dt} = k(80 - y)(60 - y) \quad \text{for some constant } k > 0. \tag{47}$$

    That is, the rate of production of C is proportional to the product of the remaining amounts of the reactants A and B.

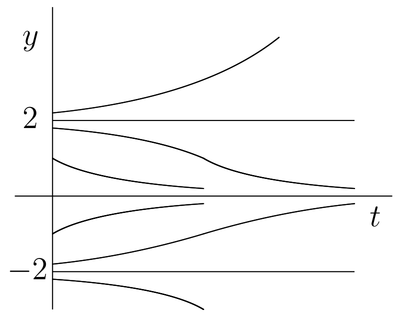

Figure 3: Possible solutions of (46)

(a) Sketch some possible solutions to the equation.

***Solution**:* Set $f(y) = k(80 - y)(60 - y)$, or $f(y) = k(y - 60)(y - 80)$, so that the differential equation in (47) has two equilibrium solutions

$$\overline{y}_1 = 60 \quad \text{and} \quad \overline{y}_2 = 80.$$

In order to determine the the stability properties of the equilibrium solutions, we first compute

$$f'(y) = k(y - 80) + k(y - 60), \tag{48}$$

where we have applied the product rule. Using (48), we compute

$$f'(60) = -20k < 0,$$

so that $\overline{y}_1 = 60$ is asymptotically stable by the principle of linearized stability; similarly, using (48) again,we compute

$$f'(80) = 20k > 0,$$

so that $\overline{y}_2 = 80$ is unstable by the principle of linearized stability.

Using the qualitative information provided by the principle of linearized stability, we obtain the sketches shown in Figure 4. □

(b) Use separation of variables to solve the above differential equation assuming that $y = 0$ when $t = 0$.

***Solution**:* Using separation of variables, we obtain

$$\int \frac{1}{(y - 80)(y - 60)}\, dy = \int k\, dt. \tag{49}$$

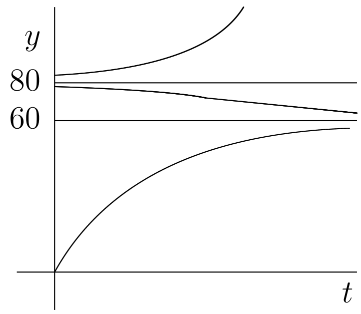

Figure 4: Possible Solutions to the equation in (47)

In order to evaluate the integral on the left–hand side of (49), we decompose the integrand by means of partial fractions as

$$\frac{1}{(y-80)(y-60)} = \frac{A}{y-80} + \frac{B}{y-60}, \tag{50}$$

where the constants A and B are to be determined. Once A and B are determined, the integral on the left–hand side of (49) can be evaluated by virtue of (50) to obtain

$$\int \frac{1}{(y-80)(y-60)}\, dy = A \ln|y-80| + B\ln|y-60| + c, \tag{51}$$

for arbitrary constant c.
In order to determine A and B, multiply on both sides of the equation in (50) by $(y-80)(y-60)$ to obtain

$$1 = A(y-60) + B(y-80),$$

or

$$0y + 1 = (A+B)y - 60A - 80B. \tag{52}$$

Equating corresponding coefficients for the polynomials on the each side of (52) yields the system

$$\left\{ \begin{array}{rcl} A+B & = & 0 \\ -60A - 80B & = & 1. \end{array} \right. \tag{53}$$

Solving the system in (53) yields

$$A = \frac{1}{20} \quad \text{and} \quad B = -\frac{1}{20}. \tag{54}$$

Substituting the values for A and B in (54) into (51) yields the left–hand side of (49) so that, integrating both sides of (49),

$$\frac{1}{20}\ln|y-80| - \frac{1}{20}\ln|y-60| = kt + c_1, \tag{55}$$

for arbitrary constant c_1. Multiply on both sides of (55) by 20 and simplify to obtain

$$\ln\left(\frac{|y-80|}{|y-60|}\right) = 20kt + c_2, \tag{56}$$

for arbitrary constant c_2. Taking the exponential on both sides of the equation in (56) and using continuity, we obtain

$$\frac{y-80}{y-60} = c\, e^{20kt}, \tag{57}$$

for arbitrary constant c.

Using the initial condition $y(0) = 0$, we obtain from (57) that

$$c = \frac{4}{3}. \tag{58}$$

Substituting the value of c in (58) into (57) and solving for y in (57) yields

$$y(t) = \frac{240(e^{20kt}-1)}{4e^{20kt}-3}, \quad \text{for } t \geqslant 0,$$

or

$$y(t) = \frac{240(1-e^{-20kt})}{4-3e^{-20kt}}, \quad \text{for } t \geqslant 0. \tag{59}$$

□

(c) In part (b), assume also that there are 20 kilograms of the reaction product 10 minutes after the onset of the reaction. How much reaction product is present 5 minutes later?

Solution: Given that $y(10) = 20$ we get from (57) and (58) that

$$\frac{3}{2} = \frac{4}{3}\cdot e^{200k},$$

which can be solved for k to yield

$$k = \frac{1}{200}\ln(9/8) \doteq 5.9\times 10^{-4}. \tag{60}$$

Using the expression for $y(t)$ in (59) and the estimate for k in (60) we obtain that

$$y(15) \doteq 26.2 \ \text{ kilograms.}$$

□