Solutions to Exam 2

1. In this problem you will solve the linear, first–order differential equation

$$\frac{dy}{dt} = -y + t. \tag{1}$$

(a) Use integration by parts to evaluate the integral $\int \tau e^{\tau}\, d\tau$.

Solution: Set

$$\begin{aligned} u = \tau &\quad \text{and} \quad dv = e^{\tau}\, d\tau \\ \text{then,}\ du = d\tau &\quad \text{and} \quad v = e^{\tau}, \end{aligned}$$

so that

$$\int \tau e^{\tau}\, d\tau = \tau e^{\tau} - \int e^{\tau}\, d\tau,$$

from which we get that

$$\int \tau e^{\tau}\, d\tau = \tau e^{\tau} - e^{\tau} + c, \tag{2}$$

where c is an arbitrary constant. □

(b) Explain why $\mu(t) = e^{t}$ is an integrating factor of the equation in (1).

Solution: Rewrite the differential equation in (1) as

$$\frac{dy}{dt} + y = t$$

and multiply by e^{t} to obtain

$$e^{t}\frac{dy}{dt} + e^{t}y = te^{t},$$

which can be written as

$$\frac{d}{dt}[e^{t}y] = te^{t}, \tag{3}$$

by virtue of the product rule. Thus, multiplying the differential equation in (1) allows one to rewrite in the form in (3), which can be integrated in order to solve for y. □

(c) Give the general solution to the equation in (1).

Solution: Integrating on both sides of (3) with respect to t yields

$$e^t y = \int t e^t \, dt. \tag{4}$$

Next, use the integration formula (2) derived in part (a) of this problem to from (4) that

$$e^t y = t e^t - e^t + c, \tag{5}$$

where c is an arbitrary constant. Solving for y in (5) we obtain

$$y(t) = t - 1 + c\, e^{-t}, \quad \text{for all } t \in \mathbf{R},$$

which is the general solution to the equation in (1). □

2. Consider the non–linear, first–order differential equation

$$\frac{dy}{dt} = (y-1)(y-2). \tag{6}$$

(a) Give the equilibrium solutions to the equation in (6) and determine their stability properties. Justify your answers.

Solution: Set $f(y) = (y-1)(y-2)$; so that the differential equation in (6) has two equilibrium solutions

$$\overline{y}_1 = 1 \quad \text{and} \quad \overline{y}_2 = 2.$$

In order to determine the the stability properties of the equilibrium solutions, we first compute

$$f'(y) = (y-1) + (y-2), \tag{7}$$

where we have applied the product rule. Using (7), we compute

$$f'(1) = -1 < 0,$$

so that $\overline{y}_1 = 1$ is asymptotically stable by the principle of linearized stability; similarly, using (7) again,we compute

$$f'(1) = 1 > 0,$$

so that $\overline{y}_2 = 2$ is unstable by the principle of linearized stability. □

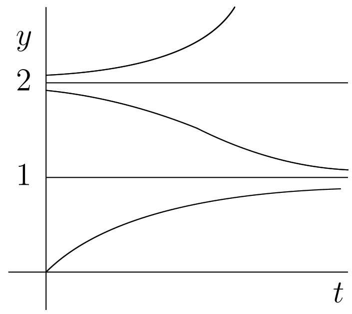

Figure 1: Possible Solutions to the equation in (6)

(b) Sketch possible solutions to the differential equation in (6).

Solution: Using the qualitative information obtained in the previous part of this problem by use of the principle of linearized stability, we obtain the sketches shown in Figure 1. □

(c) Suppose that $y = y(t)$ is a solution of (6) satisfying $y(0) = 0$. Compute $\lim_{t\to\infty} y(t)$. Justify your answer.

Solution: Examination of the sketch in Figure 1 of the solution that starts at the origin shows that

$$\lim_{t\to\infty} y(t) = 1,$$

since $y_1 = 1$ is asymptomatically stable, as shown in part (a) of this problem. □

3. In this problem you will compute the solution to the initial value problem

$$\frac{dy}{dt} = (y-1)(y-2), \qquad y(0) = 0. \tag{8}$$

(a) Determine constants, A and B, such that

$$\frac{1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2}. \tag{9}$$

Solution: Multiply on both sides of the equation in (9) by $(y-1)(y-2)$ to obtain

$$1 = A(y-2) + B(y-1),$$

or

$$0y + 1 = (A + B)y - 2A - B. \tag{10}$$

Equating corresponding coefficients for the polynomials on the each side of (10) yields the system

$$\begin{cases} A + B &= 0 \\ -2A - B &= 1. \end{cases} \tag{11}$$

Solving the system in (11) yields

$$A = -1 \quad \text{and} \quad B = 1. \tag{12}$$

□

(b) Evaluate the integral $\displaystyle\int \frac{1}{(y-1)(y-2)}\, dy$.

Solution: In view of (9) and (12), we can write

$$\frac{1}{(y-1)(y-2)} = \frac{-1}{y-1} + \frac{1}{y-2}. \tag{13}$$

Thus, integrating on both sides of (13) with respect to y yields

$$\int \frac{1}{(y-1)(y-2)}\, dy = -\ln|y-1| + \ln|y-2| + c_1,$$

for some arbitrary constant c_1, which can be written as

$$\int \frac{1}{(y-1)(y-2)}\, dy = \ln\left(\frac{|y-2|}{|y-1|}\right) + c_1. \tag{14}$$

□

(c) Use separation of variable to solve the differential equation in (8) and give its general solution.

Solution: Separation of variable yields

$$\int \frac{1}{(y-1)(y-2)}\, dy = \int dt. \tag{15}$$

Next, use the integration formula in (14) to evaluate the integral on the left–hand side of the equation in (15) to obtain

$$\ln\left(\frac{|y-2|}{|y-1|}\right) = t + c_2, \tag{16}$$

for some arbitrary constant c_2.
Taking the exponential on both sides of the equation in (16) and using continuity, we obtain

$$\frac{y-2}{y-1} = c\ e^t, \tag{17}$$

for arbitrary constant c. We can now solve (17) for y to obtain the general solution for the differential equation in (8):

$$y(t) = \frac{2 - c\ e^t}{1 - c\ e^t}. \tag{18}$$

□

(d) Give a formula for the solution, $y = y(t)$, to the initial value problem (8).

Solution: Using the initial condition, $y(0) = 0$, we obtain from (17) that

$$c = 2. \tag{19}$$

Substituting the value of c in (19) into (18) then yields the formula

$$y(t) = \frac{2 - 2\ e^t}{1 - 2\ e^t}$$

for computing the solution to the initial value problem in (8). □