Solutions to Assignment #11

1. Use the fact that $\sqrt{2} = \sup\{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 < 2\}$ to prove that there exists a sequence of rational numbers, (q_n), such that

$$\lim_{n\to\infty} q_n = \sqrt{2}.$$

Proof: Write $A = \{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 < 2\}$. Then, $\sqrt{2} = \sup(A)$. Thus, for each $n \in \mathbb{N}$ there exists $q_n \in A$ such that

$$\sqrt{2} - \frac{1}{n} < q_n < \sqrt{2}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$, it follows by the Squeeze Theorem for sequences that the limit of (q_n) exists and

$$\lim_{n\to\infty} q_n = \sqrt{2}.$$

□

2. Let (ε_n) denote a sequence of positive numbers which converges to 0. Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$. Assume there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x| \leqslant \varepsilon_n \quad \text{for all} \quad n \geqslant N_1.$$

Prove that (x_n) converges to x.

Proof: Assume there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x| \leqslant \varepsilon_n \quad \text{for all} \quad n \geqslant N_1, \tag{1}$$

where $\varepsilon_n > 0$ for all n and $\displaystyle\lim_{n\to\infty} \varepsilon_n = 0$.

Let $\varepsilon > 0$ be given. Then, there exists $N_2 \in \mathbb{N}$ such that

$$n \geqslant N_2 \Rightarrow \varepsilon_n < \varepsilon.$$

It then follows form (1) that, for $N = \max\{N_1, N_2\}$,

$$n \geqslant N \Rightarrow |x_n - x| < \varepsilon,$$

which shows that (x_n) converges to x. □

3. Let $x_n = \dfrac{1}{n!}$ for $n \in \mathbb{N}$. Prove that the sequence (x_n) converges to 0.

   Proof: First we prove by induction on n that $n! \geqslant n$ for all $n \in \mathbb{N}$. Note that $1! = 1$ and so the result is true for the base step in the induction. Next, assume that $n! \geqslant n$ and consider

   $$(n+1)! = (n+1)n! = n \cdot n! + n!.$$

   By the inductive hypothesis,

   $$(n+1)! \geqslant n \cdot n + n \geqslant n + 1.$$

   We then have that

   $$0 < \frac{1}{n!} \leqslant \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

   It then follows by the Squeeze Theorem, or by the result in the previous problem that

   $$\lim_{n\to\infty} \frac{1}{n!} = 0.$$

   □

4. Let (x_n) be a sequence of real numbers converging to $a \neq 0$. Prove that there exists $N \in \mathbb{N}$ such that

   $$n \geqslant N \Rightarrow |x_n| > \frac{|a|}{2}.$$

   Proof. Assume that $\lim\limits_{n\to\infty} x_n = a$, where $a \neq 0$. Put $\varepsilon = \dfrac{|a|}{2}$. Then, $\varepsilon > 0$ and so, by the definition of convergence, there exists $N \in \mathbb{N}$ such that

   $$n \geqslant N \Rightarrow |x_n - a| < \varepsilon = \frac{|a|}{2}.$$

   Now, by the triangle inequality

   $$|a| = |a - x_n + x_n| \leqslant |x_n - a| + |x_n|, \quad \text{for } n \geqslant N,$$

   from which we get that

   $$|a| < \frac{|a|}{2} + |x_n|, \quad \text{for } n \geqslant N.$$

   The result then follows by adding $-\dfrac{|a|}{2}$ on both sides of the previous inequality.

   □

5. Let (x_n) be a sequence of non–zero, real numbers converging to $a \neq 0$. Prove that the set $A = \left\{ \dfrac{1}{x_n} \;\middle|\; n \in \mathbb{N} \right\}$ is bounded.

   Proof: Let $x_n \neq 0$ for all $n \in \mathbb{N}$ and assume that

   $$\lim_{n\to\infty} x_n = a, \quad \text{where} \quad a \neq 0.$$

   Then, by the result in the previous problem, there exists $N \in \mathbb{N}$ such that

   $$|x_n| > \frac{|a|}{2} \qquad \text{for all} \quad n \geqslant N.$$

   We then have that

   $$\frac{1}{|x_n|} < \frac{2}{|a|} \qquad \text{for all} \quad n \geqslant N.$$

   Setting $M = \max\left\{ \dfrac{1}{|x_1|}, \dfrac{1}{|x_2|}, \ldots \dfrac{1}{|x_{N-1}|}, \dfrac{2}{|a|} \right\}$, we see that

   $$\frac{1}{|x_n|} \leqslant M \qquad \text{for all } n \in \mathbb{N};$$

   that is, the set $A = \left\{ \dfrac{1}{x_n} \;\middle|\; n \in \mathbb{N} \right\}$ is bounded. $\square$