Solutions to Assignment #12

1. Let (x_n) denote a sequence of real numbers. Prove that if $\lim_{n\to\infty} |x_n| = 0$, then (x_n) converges to 0.

   Proof: Assume that $\lim_{n\to\infty} |x_n| = 0$. Then, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that
   $$n \geqslant N \Rightarrow ||x_n| - 0| < \varepsilon.$$
   Thus,
   $$n \geqslant N \Rightarrow |x_n| < \varepsilon,$$
   or
   $$n \geqslant N \Rightarrow |x_n - 0| < \varepsilon,$$
   which shows that
   $$\lim_{n\to\infty} x_n = 0.$$
   Hence, (x_n) converges to 0. $\square$

2. Let $x_n = \dfrac{(-1)^{n+1}}{\sqrt{n}}$ for all $n \in \mathbb{N}$. Prove that (x_n) converges to 0.

   Proof: We will prove that $(|x_n|)$ converges to 0; thus, by the result of Problem 1, we will have prove that (x_n) converges to 0.

   Observe that $|x_n| = \dfrac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$. Thus, we will prove that
   $$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0. \tag{1}$$
   Let $\varepsilon > 0$ be given; then, $\varepsilon^2 > 0$. Thus, by the Archimedean Property, there exists $N \in \mathbb{N}$ such that
   $$\frac{1}{N} < \varepsilon^2.$$
   Thus, $n \geqslant N$ implies that
   $$0 < \frac{1}{\sqrt{n}} \leqslant \frac{1}{\sqrt{N}} < \varepsilon.$$
   Hence,
   $$n \geqslant N \Rightarrow \left| \frac{1}{\sqrt{n}} - 0 \right| < \varepsilon.$$
   This completes the proof of the limit in (1). $\square$

3. Let (x_n) denote a sequence of real numbers.

   (a) Prove that if (x_n) converges then $(|x_n|)$ converges.

   Proof: Assume that (x_n) converges to x. We show that $(|x_n|)$ converges to $|x|$.
   Let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that

   $$n \geqslant N \Rightarrow |x_n - x| < \varepsilon.$$

   Then, by the triangle inequality,

   $$n \geqslant N \Rightarrow ||x_n| - |x|| \leqslant |x_n - x| < \varepsilon,$$

   which shows that

   $$\lim_{n\to\infty} |x_n| = |x|.$$

   $\square$

   (b) Show that the converse of the statement in part (a) is not true.
   Solution*:* Let $x_n = (-1)^n$ for $n \in \mathbb{N}$ and observe that $|x_n| = 1$ for all n, so that $(|x_n|)$ converges to 1. However, (x_n) does not converge. $\square$

4. Let (x_n) and (y_n) denote two convergent sequences. Suppose there exists some $N_1 \in \mathbb{N}$ such that

   $$n \geqslant N_1 \Rightarrow x_n \leqslant y_n.$$

   Prove that

   $$\lim_{n\to\infty} x_n \leqslant \lim_{n\to\infty} y_n. \tag{2}$$

   Proof: Let $\lim\limits_{n\to\infty} x_n = a$ and $\lim\limits_{n\to\infty} y_n = b$. For $\varepsilon > 0$ arbitrary, there exist n_2 and n_3 in $\mathbb{N}$ such that

   $$n \geqslant n_2 \Rightarrow a - \frac{\varepsilon}{2} < x_n < a + \frac{\varepsilon}{2},$$

   and

   $$n \geqslant n_3 \Rightarrow b - \frac{\varepsilon}{2} < y_n < b + \frac{\varepsilon}{2}.$$

   Then, for $n \geqslant \max\{n_1, n_2, n_3\}$,

   $$a - \frac{\varepsilon}{2} < x_n \leqslant y_n < b + \frac{\varepsilon}{2}$$

   Thus, $a < b + \varepsilon$, for arbitrary $\varepsilon > 0$. Hence, $a \leqslant b$, which is the inequality in (2). $\square$

5. Let (x_n) and (y_n) denote sequences of real numbers. Determine whether the following statements are true or false. If false, provide a counterexample. If true provide an argument to establish the statement as true.

   (a) If (x_n) converges and $(x_n \cdot y_n)$ converges, then (y_n) converges.

   ***Solution**:* The statement is false. Consider

   $$x_n = \frac{1}{n} \text{ and } \; y_n = (-1)^n,$$

   for all $n \in \mathbb{N}$. Then, $(x_n y_n)$ converges to 0, (x_n) converges to 0, but (y_n) does not converge. $\square$

   (b) If (x_n) converges and $(x_n + y_n)$ converges, then (y_n) converges.

   ***Solution**:* This statement is true.

   Proof: Assume that (x_n) and $(x_n + y_n)$ converge. Then, since

   $$y_n = (x_n + y_n) + (-x_n) \quad \text{ for all } n \in \mathbb{N},$$

   (y_n) is the sum of two convergent series. Therefore, it converges. $\square$

$\square$