Solutions to Assignment #3

1. Let x denote a real number satisfying $x^2 = x$. Prove that either $x = 0$ or $x = 1$. (Note that $x^2 = xx$.)

   Proof: Let $x \in \mathbb{R}$ and assume that $x^2 = x$ and $x \neq 0$. Subtracting the additive inverse of x, namely $-x$, on both sides we obtain that

   $$x^2 - x = 0,$$

   or

   $$x(x-1) = 0, \tag{1}$$

   where we have used the distributive property (Axiom (F_{10}) in Handout #2). Since we are assuming that $x \neq 0$, it follows from Axiom (F_9) that there exists $x^{-1} \in \mathbb{R}$ such that

   $$x^{-1}x = 1.$$

   Multiplying on the left by x^{-1} on both sides of equation (1) we obtain

   $$x^{-1}[x(x-1)] = x^{-1}0,$$

   or

   $$x - 1 = 0, \tag{2}$$

   were we have used Axioms (F_7), (F_9), (F_8) and the fact that $a0 = 0$ for all $a \in \mathbb{R}$. Adding 1 on both sides of (2) yields

   $$x = 1.$$

   Thus, we have shown that $x^2 = x$ and $x \neq 0$ implies that $x = 1$, which is equivalent to $x^2 = x$ implies $x = 0$ or $x = 1$. □

2. Let $a \in \mathbb{R}$. Prove that if $a \neq 0$, then the equation

   $$ax = b$$

   has a unique solution for every $b \in \mathbb{R}$.

Proof: Let $a \in \mathbb{R}$ and assume that $a \neq 0$. Then, by Axiom (F_9), there exists $a^{-1} \in \mathbb{R}$ such that $a^{-1}a = 1$. Let $x = a^{-1}b$. Then, by Axioms (F_7), (F_9), (F_8) and (F_6),

$$ax = a(a^{-1}b) = b,$$

which shows that $x = a^{-1}b$ is a solution of the equation

$$ax = b.$$

To show that $ax = b$ has a unique solution, assume that x_1 and x_2 are two solutions of $ax = b$. Then,

$$ax_1 = b$$

and

$$ax_2 = b.$$

Consequently,

$$ax_1 = ax_2 \tag{3}$$

Multiplying both sides of equation (3) by a^{-1} yields, by Axioms (F_7), (F_9), (F_8) and (F_6),

$$x_1 = x_2,$$

which shows that $ax = b$ has at most one solution. □

3. Let $x \in \mathbb{R}$. Prove that $(-1)x$ is the additive inverse of x; that is $x + (-1)x = 0$.

   Proof. Let $x \in \mathbb{R}$. Use Axioms to compute

$$\begin{aligned} x + (-1)x &= 1x + (-1)x \\ &= (1 + (-1))x \\ &= 0x \\ &= 0, \end{aligned}$$

   where we have used the fact that $0x = 0$ for all real numbers x. □

4. Prove that, for any real number, x,

$$(-x)^2 = x^2.$$

Proof: Let $x \in \mathbb{R}$. Using the fact that $(-1)(-x) = x$, where $-x$ is the additive inverse of x, and the associative property of multiplication we find that

$$\begin{aligned} x^2 &= xx \\ &= [(-1)(-x)][(-1)(-x)] \\ &= (-1)(-1)(-x)(-x) \\ &= 1(-x)^2 \\ &= (-x)^2, \end{aligned}$$

which was to be shown. □

5. Let $a, b \in \mathbb{Q}$, where $a^2 + b^2 \neq 0$.

   (a) Explain by $a^2 - 2b^2 \neq 0$.
   Solution: Since $a^2 + b^2 \neq 0$, if $b = 0$, then $a \neq 0$ and so $a^2 - 2b^2 = a^2 \neq 0$ in this case. Thus, we may assume that $b \neq 0$. Then, if $a^2 - 2b^2 = 0$, we have that

   $$\frac{a^2}{b^2} = 2,$$

   or

   $$\left(\frac{a}{b}\right)^2 = 2,$$

   which shows that there is $q \in \mathbb{Q}$ such that $q^2 = 2$; namely, $q = \dfrac{a}{b}$, since $a, b \in \mathbb{Q}$. This is impossible. Hence, $a^2 - 2b^2 \neq 0$, if $a^2 + b^2 \neq 0$. □

   (b) Show that the multiplicative inverse of $a + b\sqrt{2}$, namely $(a + b\sqrt{2})^{-1}$, is of the form $c + d\sqrt{2}$, where $c, d \in \mathbb{Q}$. ***Solution***: Since $a^2 - 2b^2 \neq 0$, by part (a), we may define rational numbers

   $$c = \frac{a}{a^2 - 2b^2} \quad \text{and} \quad d = \frac{-b}{a^2 - 2b^2},$$

   since $a, b \in \mathbb{Q}$.

Using the distributive property we may compute

$$\begin{aligned}(a+b\sqrt{2})(c+d\sqrt{2}) &= \frac{1}{a^2-2b^2}(a+b\sqrt{2})(a-b\sqrt{2}) \\ &= \frac{1}{a^2-2b^2}(a^2-(b\sqrt{2})^2) \\ &= \frac{1}{a^2-2b^2}(a^2-2b^2) \\ &= 1,\end{aligned}$$

which shows that $c+d\sqrt{2}$ is the multiplicative inverse of $a+b\sqrt{2}$. □