Solutions to Assignment #6

1. Let $x \in \mathbb{R}$. Prove that $0 < x \leqslant 1$ implies that $x^2 \leqslant x$.

   Proof: Assume that $x > 0$ and $x \leqslant 1$. Then,

$$x \cdot x \leqslant x \cdot 1,$$

   from which the result follows. □

2. Let a and b denote real numbers. Use the triangle inequality to prove that

$$||a| - |b|| \leqslant |a - b|.$$

   Proof. Write $a = (a-b)+b$ and take absolute value on both sides of this identity to get

$$|a| = |(a - b) + b|.$$

   Applying the triangle inequality on the right–hand side we obtain that

$$|a| \leqslant |a - b| + |b|,$$

   from which we get that

$$|a| - |b| \leqslant |a - b|. \tag{1}$$

   Similar calculations show that

$$|b| - |a| \leqslant |b - a|.$$

   Thus, using the fact that $|b - a| = | - (a - b)| = |a - b|$ and multiplying the previous inequality by -1, we obtain that

$$-|a - b| \leqslant |a| - |b|. \tag{2}$$

   Combining the inequalities in (1) and (2) yields

$$-|a - b| \leqslant |a| - |b| \leqslant |a - b|,$$

   which is equivalent to $||a| - |b|| \leqslant |a - b|$. □

3. Let a and b denote **positive** real numbers. Start with the true statement

$$(a-b)^2 \geqslant 0$$

to prove the inequality

$$ab \leqslant \frac{a^2+b^2}{2}.$$

Prove that equality holds if and only if $a = b$.

***Solution**:* From the inequality

$$0 \leqslant (a-b)^2$$

we obtain

$$0 \leqslant a^2 - 2ab + b^2.$$

Adding $2ab$ to both sides of the last inequality we then have that

$$2ab \leqslant a^2 + b^2,$$

from which the result follows after dividing by 2.

Equality holds if and only if

$$(a-b)^2 = 0,$$

which is true if and only if $a - b = 0$, or $a = b$. $\square$

4. Given a real number x, denote by $\max\{x, 0\}$ the larger of x and 0. Prove that

$$\max\{x, 0\} = \frac{x + |x|}{2}.$$

***Solution**:* We consider two cases: (i) $x \geqslant 0$, and (ii) $x < 0$.

(i) If $x \geqslant 0$, then $\max\{x, 0\} = x$. On the other hand,

$$\frac{x+|x|}{2} = \frac{x+x}{2} = \frac{2x}{2} = x.$$

Thus, the equality is verified in this case.

(ii) If $x < 0$, then $\max\{x, 0\} = 0$, and

$$\frac{x+|x|}{2} = \frac{x-x}{2} = 0.$$

So, equality is verified in this case as well.

□

5. Let x and $\max\{x,0\}$ be as in the previous problem. Denote by $\min\{x,0\}$ the smaller of x and 0. Prove that

$$\min\{x,0\} = -\max\{-x,0\},$$

and use this result to derive a formula for $\min\{x,0\}$ analogous to that for $\max\{x,0\}$ proved in the previous problem.

***Solution**:* We consider two cases: (i) $x \geqslant 0$, and (ii) $x < 0$.

(i) If $x \geqslant 0$, then $\min\{x,0\} = 0$ and $-x \leqslant 0$, so that $\max\{-x,0\} = 0$. Thus, equality holds in this case.

(ii) If $x < 0$, then $\min\{x,0\} = x$, and $-x > 0$, so that

$$\max\{-x,0\} = -x.$$

Consequently, $-\max\{-x,0\} = x$, which is $\min\{x,0\}$ is this case.

We have therefore established that

$$\min\{x,0\} = -\max\{-x,0\}.$$

Using the formula for max derived in the previous problem we then have that

$$\begin{aligned} \min\{x,0\} &= -\left(\frac{-x+|-x|}{2}\right) \\ &= \frac{x-|x|}{2}, \end{aligned}$$

since $|-x| = |x|$ for all $x \in \mathbb{R}$. Hence,

$$\min\{x,0\} = \frac{x-|x|}{2}.$$

□