Solutions to Assignment #7

1. Let $a, b \in \mathbb{R}$. Prove that

$$a < b \text{ if and only if } a < \frac{a+b}{2} < b.$$

Proof: Assume that $a < b$. Then, adding a to both sides of the inequality yields

$$2a < a + b,$$

from which we get that

$$a < \frac{a+b}{2}.$$

On the other hand, adding b to both sides of $a < b$, we obtain

$$a + b < 2b,$$

which implies that

$$\frac{a+b}{2} < b.$$

Hence, $a < b$ implies that

$$a < \frac{a+b}{2} < b.$$

Conversely, assume that

$$a < \frac{a+b}{2} < b.$$

Multiplying by the positive number 2 then yields

$$2a < a + b < 2b.$$

Adding $-a$ to both sides of the first inequality gives

$$a < b.$$

□

2. Prove that between any two rational numbers there is at least one rational number.

Proof: Let p and q denote rational numbers and suppose that $p < q$. Then, by the result of Problem 1,

$$p < \frac{p+q}{2} < q,$$

where $\dfrac{p+q}{2}$ is a rational number since $\mathbb{Q}$ is a field. □

3. Prove that between any two rational numbers there are infinitely many rational numbers.

   Proof: Let p and q denote rational numbers with $p < q$. Assume, by way of contradiction, that there are only a finite number, n, of rational numbers

   $$q_1, q_2, \ldots, q_n;$$

   in other words, $q_1, q_2, \ldots, q_n$ are the only rational numbers between p and q. By trichotomy, since these rational numbers are distinct, we may assume that they are ordered as follows

   $$p < q_1 < q_2 < \cdots < q_n < q.$$

   Applying the result of Problem 2 to q_n and q, we obtain a rational number, r, such that $q_n < r < q$. Thus, there are $n + 1$ rational numbers between p and q. This is a contradiction. Thus, there are infinitely many rational numbers between p and q. □

4. Given two subsets, A and B, of real numbers, the union of A and B is the set $A \cup B$ defined by

   $$A \cup B = \{x \in \mathbb{R} \mid x \in A \text{ or } x \in B\}$$

   Assume that A and B are non–empty and bounded above. Prove that $\sup(A \cup B)$ exists and

   $$\sup(A \cup B) = \max\{\sup(A), \sup(B)\},$$

   where $\max\{\sup(A), \sup(B)\}$ denotes the largest of $\sup(A)$ and $\sup(B)$.

   Proof: Assume that A and B are non–empty and bounded above. Then, their suprema, $\sup(A)$ and $\sup(B)$, respectively, exist, by the completeness axiom. The assumption that A and B are non–empty also implies that $A \cup B$ is non–empty.

Let x be an arbitrary element in $A \cup B$. Then, either $x \in A$ or $x \in B$. If $x \in A$, the

$$x \leqslant \sup(A). \tag{1}$$

On the other hand, if $x \in B$, then

$$x \leqslant \sup(B). \tag{2}$$

Thus, if $x \in A \cup B$, it follows from (1) and (2) that

$$x \leqslant \max\{\sup(A), \sup(B)\},$$

since

$$\sup(A) \leqslant \max\{\sup(A), \sup(B)\} \text{ and } \sup(B) \leqslant \max\{\sup(A), \sup(B)\}.$$

Thus, $\max\{\sup(A), \sup(B)\}$ is an upper bound for $A \cup B$. Thus, by the completeness axiom $\sup(A \cup B)$ exists and

$$\sup(A \cup B) \leqslant \max\{\sup(A), \sup(B)\}. \tag{3}$$

Next, observe that

$$A \subseteq A \cup B$$

and

$$B \subseteq A \cup B,$$

from which we get that

$$\sup(A) \leqslant \sup(A \cup B)$$

and

$$\sup(B) \leqslant \sup(A \cup B).$$

Consequently,

$$\max\{\sup(A), \sup(B)\} \leqslant \sup(A \cup B). \tag{4}$$

Combining the inequalities in (3) and (4) yields the result. □

5. Given two subsets, A and B, of real numbers, the intersection of A and B is the set $A \cap B$ defined by

$$A \cap B = \{x \in \mathbb{R} \mid x \in A \text{ and } x \in B\}$$

Is it true that $\sup(A \cap B) = \min\{\sup(A), \sup(B)\}$?

Here, $\min\{\sup(A), \sup(B)\}$ denotes the smallest of $\sup(A)$ and $\sup(B)$.

Solution: The problem here is that the intersection of A and B might be empty. If this is the case $\sup(A \cap B)$ will not be defined. Thus, if $A \cap B = \emptyset$, the answer to the question is no.

On the other hand, if $A \cap B \neq \emptyset$, then, since

$$A \cap B \subseteq A \quad \text{and} \quad A \cap B \subseteq B,$$

$$\sup(A \cap B) \leqslant \sup(A) \quad \text{and} \quad \sup(A \cap B) \leqslant \sup(B),$$

It then follows that

$$\sup(A \cap B) \leqslant \min\{\sup(A), \sup(B)\}.$$

However, this inequality can be strict. For instance, let $A = \{0, 1\}$ and $B = \{0, 2\}$. Then, $A \cap B = \{0\}$; so $\sup(A \cap B) = 0$, $\sup(A) = 1$ and $\sup(B) = 2$. Thus,

$$\min\{\sup(A), \sup(B)\} = 1 > 0 = \sup(A \cap B).$$

Thus, in general, the answer to the question is no. □