Solutions to Assignment #8

1. Let $a, b \in \mathbb{R}$. Show that if $a < b + \dfrac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leqslant b$.

   Proof: Let $a, b \in \mathbb{R}$ and suppose that $a < b + \dfrac{1}{n}$ for all $n \in \mathbb{N}$. Assume, by way of contradiction that $a > b$. Then $a - b > 0$. By the Archimedean property, there exists a natural number, m, such that

   $$0 < \frac{1}{m} < a - b.$$

   On the other hand, by the assumption,

   $$a < b + \frac{1}{m}.$$

   Thus,

   $$a < b + (a - b),$$

   which says that $a < a$. This is a contradiction. We then conclude that $a \leqslant b$. □

2. Show that $\sup\{t \in \mathbb{R} \mid t < a\} = a$ for each $a \in \mathbb{R}$.

   Proof: Let $a \in \mathbb{R}$ and put $A = \{t \in \mathbb{R} \mid t < a\}$. Then, a is an upper bound for A. To see that A is not empty, first note that, if $a > 0$, then $0 \in A$. If $a = 0$, the $-1 < a$, so that $-1 \in A$. If $a < 0$, then, adding a to both sides of the inequality, we get that

   $$2a < a;$$

   thus, $2a \in A$ for $a < 0$. Hence, we have proved that $A \neq \emptyset$. It then follows by the completeness axiom that $\sup(A)$ exists and

   $$\sup(A) \leqslant a.$$

   If $\sup(A) < a$, it follows that

   $$\sup(A) < \frac{\sup(A) + a}{2} < a,$$

   which shows that $\dfrac{\sup(A) + a}{2} \in A$ and is bigger than the supremum of A. This is absurd. Consequently, $\sup(A) = a$. □

3. A subset, A, of the real numbers is said to be **bounded** if there exists a positive real number, M, such that

$$|a| \leqslant M \quad \text{for all} \quad a \in A.$$

Prove that A is bounded if and only if A is bounded above and below.

***Solution**:* Assume that A is bounded. Then there exists $M > 0$ such that

$$|a| \leqslant M \quad \text{for all } a \in A.$$

Thus,

$$-M \leqslant a \leqslant M \quad \text{for all } a \in A,$$

which shows that A is bounded above by M and below by $-M$.

Conversely, assume that A is bounded above and below. There A has an upper bound, u, and a lower bound ℓ. We may assume that $u > 0$ (if not, replace u by $u + 2|u| > 0$, which is also and upper bound for A). Let $M = \max\{u, |\ell|\}$. Then, for every $a \in A$,

$$a \leqslant u \leqslant M,$$

and

$$a \geqslant \ell \geqslant -|\ell|,$$

where $|\ell| \leqslant \max\{u, |\ell|\}$, so that

$$-|\ell| \geqslant -M.$$

Thus,

$$-M \leqslant a \leqslant M \quad \text{for all } a \in A,$$

which shows that A is bounded. □

4. For real numbers a and b with $a < b$, $[a, b]$ denotes the closed, bounded, interval from a to b; that is,

$$[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}.$$

Assume that $A \subseteq \mathbb{R}$ is nonempty and bounded. Prove that

$$A \subseteq [\inf(A), \sup(A)].$$

Proof: Assume that A non–empty and bounded. Then $\inf(A)$ and $\sup(A)$ exist. Thus,

$$\inf(A) \leqslant a \leqslant \sup(A) \quad \text{for all } a \in A,$$

which shows that $a \in [\inf(A), \sup(A)]$ for all $a \in A$. In other words, A is a subset of $[\inf(A), \sup(A)]$. □

5. Let A denote a nonempty and bounded subset of the real numbers. Prove that if I is a closed interval with $A \subseteq I$, then

$$[\inf(A), \sup(A)] \subseteq I.$$

Proof: Assume that $A \subset \mathbb{R}$ is nonempty and bounded, with $A \subseteq I$, where I is a closed and bounded interval. Then, $\inf(A)$ and $\sup(A)$ exist. We show that

$$[\inf(A), \sup(A)] \subseteq I.$$

Write $I = [a, b]$. Since $A \subseteq I$,

$$a \leqslant x \leqslant b \quad \text{for all } x \in A.$$

Thus, b is an upper bound for A and a is a lower bound for A. Consequently,

$$\sup(A) \leqslant b$$

and

$$a \leqslant \inf(A).$$

We then have that

$$a \leqslant \inf(A) \leqslant \sup(A) \leqslant b. \tag{1}$$

Now, if $x \in [\inf(A), \sup(A)]$, then

$$\inf(A) \leqslant x \leqslant \sup(A).$$

It then follows from (1) that

$$a \leqslant x \leqslant b,$$

which shows that $x \in [a, b]$. We have therefor shown the implication

$$x \in [\inf(A), \sup(A)] \Rightarrow x \in [a, b],$$

which is equivalent to

$$[\inf(A), \sup(A)] \subseteq I.$$

□