Solutions to Assignment #9

1. Let x denote a positive real number. Prove that $0 < z < 1$ implies that $zx < x$.

   Proof. Assume that $0 < z < 1$ and $x > 0$. Then, $1 - z > 0$. Thus, by the Order Axiom (O_3),
   $$x(1 - z) > 0,$$
   from which we get that $x - xz > 0$, by the distributive property. Hence,
   $$zx < x.$$
   □

2. Let A and B be a non–empty subsets of $\mathbb{R}$ which are bounded from above. Prove that if $\sup A < \sup B$, then there exists $b \in B$ such that b is an upper bound for A.

   Proof: Assume that $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ are nonempty and bounded above. Then, by the completeness axiom, $\sup(A)$ and $\sup(B)$ exist.

   If $\sup(A) < \sup(B)$, then there exists $b \in B$ such that
   $$\sup(A) < b,$$
   otherwise $\sup(A)$ would be an upper bound for B which is smaller that $\sup(B)$, which is impossible. Thus,
   $$x \leqslant \sup(A) < b \quad \text{for all } x \in A,$$
   which shows that b is an upper bound for A. □

3. Let A be a non–empty and bounded subset of $\mathbb{R}$. Prove that
   $$\inf(A) \leqslant \sup(A).$$

   Proof: Assume that $A \neq \emptyset$ is bounded. Then A is bounded above and below. Therefore $\inf(A)$ and $\sup(A)$ exist and
   $$\inf(A) \leqslant x \leqslant \sup(A) \quad \text{for all } x \in A.$$
   The result follows from this inequality. □

4. Let $a \in \mathbb{R}$ and define the sets

$$A = \{x \in \mathbb{R} \mid x < a\}$$

and

$$B = \{q \in \mathbb{Q} \mid q < a\}.$$

Prove that the suprema of A and B exist and

$$\sup(A) = \sup(B) = a.$$

***Solution**:* Observe that both A and B are bounded above by a. Note also that B is nonempty. In fact, by the density of the set of rational numbers in the real numbers, there exists a rational number q such that

$$a - 1 < q < a,$$

so that $q \in B$. Observe also that B is a subset of A. Thus, A is also nonempty. Consequently, by the Completeness Axiom, $\sup(A)$ and $\sup(B)$ exist. Furthermore,

$$\sup(B) \leqslant \sup(A) \leqslant a. \tag{1}$$

Suppose, by way of contradiction, that $\sup(B) < \sup(A)$. Then, by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number q such that

$$\sup(B) < q < \sup(A). \tag{2}$$

It follows from (2) and (1) that $q < a$ so that $q \in B$. However, this is in direct contradiction with the left–most inequality in (2). This contradiction shows that

$$\sup(B) = \sup(A). \tag{3}$$

We next show that $\sup(B) = a$. Arguing by contradiction again, assume, in view of (1), that $\sup(B) < a$. Invoking the density of $\mathbb{Q}$ in $\mathbb{R}$ again, we get that there exists $q \in \mathbb{Q}$ such that

$$\sup(B) < q < a. \tag{4}$$

It follows from the right–most inequality in (4) that $q \in B$, which is in contradiction with the left–most inequality in (4). This contradiction establishes that

$$\sup(B) = a. \tag{5}$$

The results in (3) and (5) together imply what we were asked to prove. $\square$

5. Use the fact that between any two distinct real numbers there is a rational number to prove the statement:

   Between any two distinct real numbers there is at least one irrational number.

   ***Solution**:* Let x and y denote distinct real numbers and assume, without loss of generality, that

$$x < y. \tag{6}$$

   We have seen in class that $\sqrt{2}$ is an irrational number. Adding $\sqrt{2}$ to both sides on the inequality in (6) we obtain

$$x + \sqrt{2} < y + \sqrt{2}.$$

   Next, use the fact that between any two distinct real numbers there is a rational number to obtain $q \in \mathbb{Q}$ such that

$$x + \sqrt{2} < q < y + \sqrt{2}. \tag{7}$$

   Adding $-\sqrt{2}$ to every term in the inequality in (7) we obtain that

$$x < q - \sqrt{2} < y,$$

   and observe that $q - \sqrt{2}$ is irrational. □