Solutions to Exam 1 (Part II)

1. Let A be a non–empty subset of $\mathbb{R}$. Prove that if u is an upper bound for A and $u \in A$, then $u = \sup A$.

   Proof: Assume that A is nonempty and that u is an upper bound for A. Then, by the completeness axiom, $\sup(A)$ exists and

   $$\sup(A) \leqslant u. \tag{1}$$

   On the other hand, since is an element of A, it follows that

   $$u \leqslant \sup(A). \tag{2}$$

   Combining (1) and (2) yields the equality

   $$u = \sup A,$$

   which was to be shown. □

2. In each of the following, show that the given set A is bounded, and compute $\sup(A)$ and $\inf(A)$.

   (a) $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$; in other words, A is the open interval $(0, 1)$.

   ***Solution**:* Observe that 0 is a lower bound of A and 1 is an upper bound. Since A is not empty, it follows from the completeness axiom that $\sup(A)$ exists and

   $$\sup(A) \leqslant 1. \tag{3}$$

   Similarly, by a consequence of the completeness axiom proved in class, $\inf(A)$ exists and

   $$\inf(A) \geqslant 0. \tag{4}$$

   We claim that

   $$\sup(A) = 1. \tag{5}$$

   Arguing by contradiction, if (5) does not hold true, then, in view of (3),

   $$\sup(A) < 1. \tag{6}$$

   Then, adding 1 to both sides of (6),

   $$\sup(A) + 1 < 2. \tag{7}$$

Dividing both sides of (7) by 2 then yields

$$\frac{\sup(A)+1}{2} < 1. \tag{8}$$

On the other hand,

$$\sup(A)+1 \geqslant \inf(A)+1 \geqslant 1, \tag{9}$$

where we have used (4). Dividing the inequality in (9) by 2 we obtain

$$\frac{\sup(A)+1}{2} \geqslant \frac{1}{2} > 0, \tag{10}$$

where we have used the fact that $\frac{1}{2} > 0$. It follows from (8) and (10) that the number

$$x = \frac{\sup(A)+1}{2} \tag{11}$$

is an element of A. However, using the assumption in (6), we obtain from the definition of x in (11) that

$$x > \frac{\sup(A)+\sup(A)}{2} = \sup(A). \tag{12}$$

Note that (12) is in direct contradiction with the fact that $x \in A$. This contradiction establishes that the claim in (5) is true.

Next, we show that

$$\inf(A) = 0. \tag{13}$$

Arguing again by contradiction, if (13) is not true, then, by virtue of (4),

$$\inf(A) > 0. \tag{14}$$

Put

$$y = \frac{\inf(A)}{2}. \tag{15}$$

Then, since $0 < \frac{1}{2} < 1$,

$$y < \inf(A) \tag{16}$$

and

$$0 < y < 1, \tag{17}$$

where we have used (14), the definition of y in (15), and the fact that $\inf(A) \leqslant \sup(A) = 1$, by (5). It follows from (17) that $y \in A$; however, this is in direct contradiction with (16). We therefore deduce that (13) is true. □

(b) $A = \left\{ \dfrac{n}{n+1} \;\middle|\; n \in \mathbb{N} \right\}$.

***Solution**:* First, observe that, for all $n \in \mathbb{N}$, $n < n+1$; so that,

$$\frac{n}{n+1} < 1, \qquad \text{for all } n \in \mathbb{N}.$$

Thus, 1 an upper bound for A; and so A is bounded above. Since, A is also nonempty, it follows from the Completeness Axiom that $\sup(A)$ exists and

$$\sup(A) \leqslant 1. \tag{18}$$

We claim that

$$\sup(A) = 1. \tag{19}$$

In order to establish (19), argue by contradiction. Thus, in view of (18), we assume that

$$\sup(A) < 1; \tag{20}$$

so that

$$\frac{1}{\sup(A)} > 1,$$

and

$$\frac{1}{\sup(A)} - 1 > 0. \tag{21}$$

It follows from (21) and the Archimedean Property that there exists a natural number m such that

$$\frac{1}{m} < \frac{1}{\sup(A)} - 1. \tag{22}$$

Rearranging the terms in (22) leads to

$$\sup(A) < \frac{m}{m+1}. \tag{23}$$

However, (23) is in contradiction with fact tat $\dfrac{m}{m+1} \in A$. Hence, (19) is established.

Next, observe that $\dfrac{1}{n} \leqslant 1$ for all $n \in \mathbb{N}$; thus,

$$1 + \frac{1}{n} \leqslant 2, \qquad \text{for all } n \in \mathbb{N},$$

from which we get that

$$\frac{1}{1+\frac{1}{n}} \geqslant \frac{1}{2}, \quad \text{for all } n \in \mathbb{N},$$

or

$$\frac{n}{n+1} \geqslant \frac{1}{2}, \quad \text{for all } n \in \mathbb{N}. \tag{24}$$

It follows from (24) that $\frac{1}{2}$ is a lower bound for A. Consequently, $\inf(A)$ exists and

$$\inf(A) \geqslant \frac{1}{2}. \tag{25}$$

Since $\frac{1}{2} \in A$, it follows that

$$\inf(A) \leqslant \frac{1}{2}. \tag{26}$$

Combining (25) and (26) yields that

$$\inf(A) = \frac{1}{2}.$$

□

3. Let $B \subseteq \mathbb{R}$ be a non–empty subset which is bounded from below and put $\ell = \inf B$. Prove that for every $n \in \mathbb{N}$ there exists $x_n \in B$ such that

$$\ell \leqslant x_n < \ell + \frac{1}{n}.$$

Proof: Assume that $B \subset \mathbb{R}$ is nonempty and bounded below. Then, $\inf(B)$ exists. Given any $n \in \mathbb{N}$, $\frac{1}{n} > 0$, since $n \geqslant 1 > 0$ for all $n \in \mathbb{N}$. It then follows that

$$\inf(B) < \inf(B) + \frac{1}{n}.$$

Thus, there exists $x_n \in B$ such that

$$\inf(B) \leqslant x_n < \inf(B) + \frac{1}{n}.$$

Otherwise, all the elements of B would be above $\inf(B) + \dfrac{1}{n}$, and $\inf(B) + \dfrac{1}{n}$ would be a lower bound for B greater than the greatest lower bound. Hence, putting $\ell = \inf B$, we have that for any $n \in \mathbb{N}$ there exists $x_n \in B$ such that

$$\ell \leqslant x_n < \ell + \frac{1}{n}.$$

□