Solutions to Review Problems for Exam #1

1. Let B denote a non–empty subset of the real numbers which is bounded below. Define
$$A = \{x \in \mathbb{R} \mid x \text{ is a lower bound for } B\}.$$
Prove that A is non–empty and bounded above, and that $\sup A = \inf B$.

   ***Solution**:* Since B is bounded below, there exists $\ell \in \mathbb{R}$ such that ℓ is a lower bound for B. Hence, $\ell \in A$ and, therefore, A is not empty.

   Next, use the assumption that B is non–empty to conclude that there exists $b \in B$. Then, for any lower bound, ℓ, of B,
$$\ell \leqslant b.$$
   Hence, b is an upper bound for A.

   Thus, we have shown that A is non–empty and bounded above. Therefore, by the Completeness Axiom, $\sup(A)$ exists.

   We show next that $\sup(A)$ is the infimum of B.

   First we show that $\sup(A)$ is a lower bound for B. Let $\ell \in A$, then
$$\ell \leqslant b \quad \text{for every} \quad b \in B.$$
   Thus, every $b \in B$ is an upper bound for A. Consequently,
$$\sup(A) \leqslant b \quad \text{for every} \quad b \in B.$$
   Hence, $\sup(A)$ is a lower bound for B.

   Next, let c be a lower bound for B. Then $c \in A$ and therefore
$$c \leqslant \sup(A);$$
   that is, $\sup(A)$ is greater or equal to any lower bound for B. In other words,
$$\sup(A) = \inf(B),$$
   which was to be shown. □

2. Prove that, for any real number, x,
$$|x^2| = |x|^2 = x^2.$$

Proof: Compute

$$\begin{aligned} |x^2| &= |xx| \\ &= |x||x| \\ &= |x|^2. \end{aligned}$$

On the other hand, by the definition of the absolute value function,

$$|x^2| = x^2,$$

since $x^2 \geqslant 0$. It then follows that $|x|^2 = x^2$, and the proof is now complete. $\square$

3. Let $a, b, c \in \mathbb{R}$ with $c > 0$. Show that $|a - b| < c$ if and only if $b - c < a < b + c$.

   ***Solution**:* $|a - b| < c$ if and only if $-c < a - b < c$, which is true if and only if

   $$b - c < a < b + c,$$

   where we have added b to each part of the inequality. $\square$

4. Let $a, b \in \mathbb{R}$. Show that if $a < x$ for all $x > b$, then $a \leqslant b$.

   Proof: Assume, by way of contradiction, that $a < x$ for all $x > b$ and $a > b$. It then follows that $a < a$, which is absurd. Hence, $a < x$ for all $x > b$ implies that $a \leqslant b$. $\square$

5. Show that the set $A = \{1/n \mid n \in \mathbb{N}\}$ is bounded above and below, and give its supremum and infimum.

   ***Solution**:* Observe that $\dfrac{1}{n} \leqslant 1$ for all $n \in \mathbb{N}$. It then follows that 1 is an upper bound for A. Since, $A \neq \emptyset$, $\sup(A)$ exists and

   $$\sup(A) \leqslant 1.$$

   To see that $\sup(A) = 1$, observe that $1 \in A$ and therefore $1 \leqslant \sup(A)$.

   Next, observe that $n > 0$ for all $n \in \mathbb{N}$. It then follows that $n^{-1} > 0$ for all n in $\mathbb{N}$. Thus, 0 is a lower bound for A. Consequently, the infimum of A exists and

   $$0 \leqslant \inf(A).$$

To see that $\inf(A) = 0$, assume to the contrary that $\inf(A) > 0$; then $\dfrac{1}{\inf(A)} > 0$. Since $\mathbb{N}$ is unbounded, there exists a natural number, n, such that

$$n > \frac{1}{\inf(A)}.$$

It then follows that

$$\frac{1}{n} < \inf(A),$$

which is impossible since $\dfrac{1}{n} \in A$. Thus, $\inf(A) = 0$. $\square$

6. Let $A = \{n + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$. Compute $\sup A$ and $\inf A$, if they exist.

   ***Solution**:* First note that, since

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \leqslant 1,$$

for all $n \in \mathbb{N}$, it follows that

$$n + \frac{(-1)^n}{n} \geqslant n - \left|\frac{(-1)^n}{n}\right| \geqslant n - 1 \qquad (1)$$

for all $n \in \mathbb{N}$. Consequently, the set A is not bounded since $\mathbb{N}$ is unbounded. Therefore, $\sup(A)$ does not exist.

On the other hand, it follows from the inequality in (1) that

$$n + \frac{(-1)^n}{n} \geqslant 0$$

for all $n \in \mathbb{N}$. Thus, 0 is a lower bound for A. Therefore, since A is not empty, $\inf(A)$ exists and

$$\inf(A) \geqslant 0.$$

To see that $\inf(A) = 0$, note that $0 \in A$. $\square$

7. Let $A = \{1/n \mid n \in \mathbb{N} \text{ and } n \text{ is prime}\}$. Compute $\sup A$ and $\inf A$, if they exist.

   ***Solution**:* Since $n = 2$ is the smallest prime, it follows that $n \geqslant 2$ for all $n \in \mathbb{N}$ which are prime. It then follows that

$$a \leqslant \frac{1}{2} \qquad \text{for all} \quad a \in A.$$

Thus, $\dfrac{1}{2}$ is an upper bound for A. Hence, since A is non–empty, $\sup(A)$ exists and
$$\sup(A) \leqslant \frac{1}{2}.$$
In fact, $\sup(A) = \dfrac{1}{2}$ since $\dfrac{1}{2} \in A$.

Next, note that, by definition, prime numbers are positive. Consequently, $a > 0$ for all $a \in A$ and therefore 0 is a lower bound for A. Thus, $\inf(A)$ exists and
$$\inf(A) \geqslant 0.$$
To see that $\inf(A) = 0$, argue by contradiction. If $\inf(A) > 0$, then $\dfrac{1}{\inf(A)} > 0$, and so, since the set of primes is unbounded, there exists a prime number, p, with
$$\frac{1}{\inf(A)} < p,$$
from which we get that
$$\inf(A) > \frac{1}{p},$$
which is impossible since $\dfrac{1}{p} \in A$. Therefore, $\inf(A) = 0$. $\square$

8. Let A denote a subset of $\mathbb{R}$. Give the negation of the statement: "A is bounded above."

   Solution*:* First, translate the statement "A is bounded above" into
   $$\exists u \in \mathbb{R} \text{ such that } (\forall a \in A)\ a \leqslant u.$$
   Thus, the negation of the statement reads
   $$(\forall u \in \mathbb{R})\ (\exists a \in A) \text{ such that } a > u.$$
   In other words, "for every real number, u, it is possible to find an element of A which is bigger than u." $\square$

9. Let $A \subseteq \mathbb{R}$ be non–empty and bounded from above. Put $s = \sup A$. Prove that for every $n \in \mathbb{N}$ there exists $x_n \in A$ such that
   $$s - \frac{1}{n} < x_n \leqslant s.$$

Proof: Note that for all $n \in \mathbb{N}$, $\dfrac{1}{n} > 0$. Thus,

$$s - \frac{1}{n} < s.$$

Thus, for each $n \in \mathbb{N}$, it is possible to find an element of A, call it x_n, such that

$$s - \frac{1}{n} < x_n;$$

otherwise,

$$x \leqslant s - \frac{1}{n} \qquad \text{for all} \quad x \in A,$$

which would say that $s - \dfrac{1}{n}$ is an upper bound of A, smaller than $\sup(A)$. This is impossible. Hence, for every $n \in \mathbb{N}$ there exists $x_n \in A$ such that

$$s - \frac{1}{n} < x_n \leqslant s.$$

□

10. What can you say about a non–empty subset, A, of real numbers for which $\sup A = \inf A$.

    ***Solution**:* Assume that $A \subseteq \mathbb{R}$ is non–empty with $\sup(A) = \inf(A)$.

    Let a denote any element in A. Then,

    $$\sup(A) = \inf(A) \leqslant a \leqslant \sup(A),$$

    which shows that $a = \sup(A)$. Thus,

    $$A = \{\sup(A)\};$$

    in other words, A consists of a single element, $\sup(A)$. □