Problem Set #3: Completeness Axiom (Part I)

Read: Chapter 5 on *Upper Bounds and Suprema,* pp. 80–85, in Michael J. Schramm' book: "Introduction to Real Analysis."

Definitions and Notation.
Given a non–empty subset A of $\mathbb{R}$, if A is bounded from above, then the least upper bound of A exists by the Completeness Axiom. We shall denote the least upper bound by $\sup A$ and refer to it also as the **supremum** of the set A.
Let B be a non–empty subset of $\mathbb{R}$. We say that B is **bounded from below** if there exists $s \in \mathbb{R}$ such that $s \leq x$ for all $x \in B$. We call s a **lower bound** for B. A lower bound b for B with the property that $s \leq b$ for any lower bound s of B is called the **greatest lower bound** for B. If it exists, we denote the greatest lower bound of B by $\inf B$ and refer to it also as the **infimum** of B.

Problems:

1. Let A be a non–empty subset of $\mathbb{R}$. Prove that if $\inf A$ and $\sup A$ exist, then they are unique.

2. Let B be a non–empty subset of $\mathbb{R}$. Denote by $-B$ the set $\{x \in \mathbb{R} \mid -x \in B\}$.

   (a) Prove that if B is bounded from below, then $-B$ is bounded from above. Consequently, $\sup(-B)$ exists.

   (b) Prove that if B is bounded from below, then B has a greatest lower bound and that
   $$\inf B = -\sup(-B).$$

3. Let A and B be subsets of $\mathbb{R}$ and define $A + B$ to be the set
   $$\{z \in \mathbb{R} \mid z = x + y, \text{ where } x \in A \text{ and } y \in B\}.$$

   (a) Prove that if A and B are non–empty and bounded from above, then so is $A + B$ and that
   $$\sup(A + B) \leqslant \sup A + \sup B.$$

   (b) Prove that if A and B are non–empty and bounded from below, then so is $A + B$ and that
   $$\inf(A + B) \geqslant \inf A + \inf B.$$

4. Let A and B be non–empty subsets of $\mathbb{R}$. Prove the following statements.

   (a) If $A \subseteq B$ and B is bounded from above, then $\sup A \leqslant \sup B$.

   (b) If $A \subseteq B$ and B is bounded from below, then $\inf B \leqslant \inf A$.

5. For a subset A of the real numbers and $c \in \mathbb{R}$, define: (i) $A + c = \{y \in \mathbb{R} \mid y = x + c \text{ where } x \in A\}$, and (ii) $cA = \{y \in \mathbb{R} \mid y = cx \text{ where } x \in A\}$. Prove the following statements.

   (a) Suppose that A is non–empty and bounded above, then
   $$\sup(A + c) = \sup A + c.$$

   (b) Suppose that A is non–empty and bounded above. If $c > 0$, then
   $$\sup(cA) = c \sup A.$$

   (c) Suppose that A is non–empty and bounded above. If $c < 0$, then
   $$\inf(cA) = c \sup A.$$

6. Let A be a non–empty subset of $\mathbb{R}$ which is bounded from above. Then, $s = \sup A$ if and only if

   (i) for each $\varepsilon > 0$, if $x \in A$ then $x < s + \varepsilon$, and

   (ii) for each $\varepsilon > 0$ there exists $x \in A$ such that $s - \varepsilon < x$.

7. State and prove a result analogous to that of problem (7) for a non–empty subset B of $\mathbb{R}$ which is bounded from below.

8. Let A be a non–empty subset of $\mathbb{R}$. Prove that if u is an upper bound for A and $u \in A$, then $u = \sup A$.

9. Let A and B be a non–empty subsets of $\mathbb{R}$ which are bounded from above. Prove that if $\sup A < \sup B$, then there exists $b \in B$ such that b is an upper bound for A.