Problem Set #6: Convergence of Sequences

Read: Chapter 5 on *Upper Bounds and Suprema,* pp. 80–85, in Michael J. Schramm's book: "Introduction to Real Analysis."

Read: Section 9.2 on *Convergence,* pp. 147–150, in Michael J. Schramm's book: "Introduction to Real Analysis."

Definitions and Notation.

- *Monotone Sequences.* A sequence (x_n) is said to be **increasing** if $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$; the sequence (x_n) is said to be **decreasing** if $x_n \geqslant x_{n+1}$ for all $n \in \mathbb{N}$. A sequence (x_n) is said to be **monotone** if it is either increasing or decreasing.

- *Bounded Sequences.* The sequence (x_n) is said to be **bounded** if there exists a real number $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$.

- *Subsequences.* Let (x_n) be a sequence of real numbers, and let $(n_j) = (n_1, n_2, n_3, \ldots)$ be an increasing sequence of distinct natural numbers, then the sequence (x_{n_j}) is called a **subsequence** of (x_n).

- *Cauchy Sequences.* A sequence (x_n) of real numbers is said to be a **Cauchy sequence** iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$, which depends on ε, such that

$$n, m \geqslant N \implies |x_n - x_m| < \varepsilon.$$

Problems:

1. In this problem, we show that any sequence of real numbers has a monotone subsequence.

   (a) Let B denote an infinite subset of $\mathbb{N}$. Prove that we can write

   $$B = \{n_k \mid k \in \mathbb{N}\},$$

   where

   $$n_1 < n_2 < n_3 < \cdots$$

   Furthermore, B is unbounded and, for every $M > 0$, there exists $K \in \mathbb{N}$ such that

   $$k \geqslant K \Rightarrow n_k > M;$$

   we write: $n_k \to \infty$ as $k \to \infty$.

(b) Let (x_n) denote a sequence of real numbers and consider the following subset of the natural numbers

$$B = \{n \in \mathbb{N} \mid x_n \geqslant x_m \text{ for all } m \geqslant n\}.$$

(i) Prove that if B is finite or empty, then there exist $n_1, n_2, \ldots, n_k, \ldots$ such that

$$n_1 < n_2 < \cdots < n_k < \cdots$$

and

$$x_{n_k} < x_{n_{k+1}}$$

for all $k \in \mathbb{N}$.

(ii) Prove that if B is infinite, then there exist $n_1, n_2, \ldots, n_k, \ldots$ such that

$$n_1 < n_2 < n_3 < \cdots,$$

where $n_k \to \infty$ as $k \to \infty$, and

$$x_{n_{k+1}} \leqslant x_{n_k} \quad \text{for all } k \in \mathbb{N}.$$

(c) Deduce that every sequence of real numbers has a monotone subsequence.

2. Let (x_n) be a bounded sequence of real numbers. Prove that (x_n) has a subsequence which converges.

3. Let (x_n) be a sequence of real numbers. Prove that if (x_n) converges, then it is a Cauchy sequence.

4. Prove that every Cauchy sequence of real numbers must be bounded.

5. Let (x_n) be a Cauchy sequence of real numbers. Prove that if (x_n) has a subsequence that converges to $x \in \mathbb{R}$, then $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = x$.

6. *Cauchy's Criterion for Convergence.* Prove that every Cauchy sequence of real numbers must converge.