Assignment #15

Due on Friday, November 30, 2012

Read Section 6.3, *Interpretations of the Derivative,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Sections 2–1, 2–2, 2–3, 2–4 and 2–5, pp. 47–54, in *The Calculus Primer* by William L. Schaaf.

Background and Definitions

- *(Linear Approximation of a Differentiable Function).* Let f denote a real valued function defined in an open interval, I, of the real line containing a point a. Assume that f is differentiable at a. The **linear approximation** to f at a, denoted by $L_f(a;x)$, is defined by

  $$L_f(a;x) = f(a) + f'(a)(x-a), \quad \text{for } x \in \mathbb{R}.$$

  The fact that f is differentiable at a implies that

  $$f(x) = L_f(a;x) + E_f(a;x), \quad \text{for } t \in I, \text{ where } \lim_{x\to a} \frac{|E_f(a;x)|}{|x-a|} = 0,$$

  where $E_f(a;x) = f(x) - f(a) - f'(a)(x-a)$, for $x \in I$, is the **error** term in the approximation $f(x) \approx f(a) + f'(a)(x-a)$, for x in I very close to a.

- *(Tangent Line to a Curve in the Plane).* Let f denote a real valued function defined in an open interval, I, of the real line containing a point a. Assume that f is differentiable at a. Then, the derivative of f at a gives the slope of the tangent line to the graph of $y = f(x)$ in the xy–plane over the interval I. The equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ is $y = f(a) + f'(a)(x-a)$.

Do the following problems

1. Let f denote a continuous function defined on some open interval that contains a. Suppose that $L(x) = m(x-a)+b$ is the best linear function that approximates f near a in the sense that

   $$\lim_{x\to a} \frac{|f(x) - L(x)|}{|x-a|} = 0. \tag{1}$$

   (a) Determine the value of b in the definition of $L(x)$.

(b) Show that if (1) holds true, then f is differentiable at a and determine the value of m in the definition of $L(x)$.

2. Let $f(x) = \dfrac{1}{x}$, for $x > 0$.

   (a) Give the equation to the tangent line to the graph of of $y = f(x)$ at the point $(1, 1)$.

   (b) Sketch the graphs of $y = f(x)$ and its tangent line at $(1, 1)$ and determine the point on the x–axis where the tangent line intersects that axis.

3. Let $f(x) = \sqrt{x}$, for $x \geqslant 0$.

   (a) Give the linear approximation to f at $a = 1$.

   (b) Use the linear approximation to f near 1 to estimate $\sqrt{0.98}$. Compare your estimate to that given by a calculator.

4. Let $f(x) = \cos x$, for $x \in \mathbb{R}$.

   (a) Give the linear approximation to f at $a = \dfrac{\pi}{3}$.

   (b) Use the linear approximation to f near $\dfrac{\pi}{3}$ to estimate $\cos(61^\circ)$. Compare your estimate to that given by a calculator.

5. Let $f(x) = x^{2/3}$ for all $x \in \mathbb{R}$. Explain why the tangent line to the graph of $y = f(x)$ at $(0, 0)$ cannot be defined.