Assignment #2

Due on Monday, September 17, 2012

Read Chapter 3, *The Concept of Limit,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions

- *The Binomial Theorem.*

  Let a and b denote real numbers and n a positive integer. The formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \tag{1}$$

  where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are called binomial coefficients, gives the expansion of the binomial $a+b$ raised to the n^{th} power. The symbol $n!$ denotes the factorial of n; namely,

$$n! = n(n-1)(n-2)\cdots 2\cdot 1.$$

- *The Squeeze Lemma.*

  Let (a_n), (b_n) and (c_n) be three sequences. Suppose that there exists a positive integer n_1 such that

$$a_n \leqslant b_n \leqslant c_n, \quad \text{for all } n \geqslant n_1.$$

  Assume in addition that the sequences (a_n) and (c_n) converge to the same limit ℓ; that is, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = \ell$. Then, the sequence (b_n) converges to ℓ; that is,

$$\lim_{n\to\infty} b_n = \ell.$$

Do the following problems

1. *Factorials.* The factorial of a positive integer, n, is defined by

$$n! = n(n-1)(n-2)\cdots 2\cdot 1.$$

   The factorial of 0 is defined to be $0! = 1$.

   (a) Explain why $(n+1)! = (n+1)\cdot n!$.

   (b) Show that $(n+1)! \geqslant 2^n$ for all $n \geqslant 1$.

2. *Binomial Coefficients.*

   (a) Compute $\binom{3}{k}$ for $k = 0, 1, 2, 3$.

   (b) Use the Binomial Theorem in (1) to show that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n. \tag{2}$$

3. *The limit of the sequence* $\left(\frac{1}{2^n}\right)$.

   (a) Use the result in (2) to deduce that

$$2^n \geqslant n + 1, \quad \text{for all } n = 1, 2, 3, \ldots$$

   (b) Use the Squeeze Lemma and the fact that $\lim_{n\to\infty} \frac{1}{n+1} = 0$ to deduce that

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

4. Show that the sequence $\left(\frac{1}{n^2}\right)$ converges and compute its limit.

   Suggestion: Observe that $n^2 \geqslant n$ for all $n \geqslant 1$ and apply the Squeeze Lemma.

5. Use the limit facts discussed in the lecture notes and the results of the previous problems (if necessary) to compute the following limits.

   (a) $\lim_{n\to\infty} \frac{1 + n - 2n^2}{n^2}$.

   (b) $\lim_{n\to\infty} \frac{n^2 + 1}{2 - n + 3n^2}$.