Assignment #8

Due on Wednesday, October 24, 2012

Read Section 5.1, *The Area Problem,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Sections 15-1, 15-2 and 15-3, pp. 318–321, in *The Calculus Primer* by William L. Schaaf.

Background and Definitions

- **Theorem** *(Existence of the Area Function).* Let f denote a piecewise continuous function defined on an interval containing a point a. Then, for any x in that interval, $A_f(a:x)$ exists. Furthermore, for the case $x > a$, $A_f(a:x)$ can be obtained as a limit as n tends to infinity of the sequence of sums of the form

$$\sum_{k=1}^{n} f(\tau_k)(t_k - t_{k-1}), \tag{1}$$

where

$$[t_o, t_1], [t_1, t_2], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n], \qquad \text{with } t_o = a \text{ and } t_n = x, \tag{2}$$

is any subdivision of the interval $[a, x]$ with the property that the largest length of the intervals in (2) tends to 0 as n tends to infinity, and τ_k is any point in the subinterval $[t_{k-1}, t_k]$.

- **Notation** *(Riemann Sums and the Riemann Integral).* The sums in (1) are called Riemann sums. The area function, $A_f(a;x)$, when it exists is usually denoted by the the symbol $\displaystyle\int_a^x f(t)\ dt$, and is referred to as the Riemann integral of f over the interval $[a, x]$.

Do the following problems

1. Let I denote an interval where f is defined and piecewise continuous. Explain why the following properties of the area function are true.

   (a) For any real number a in the interval I, $A_f(a;a) = 0$.

   (b) For any real number a b and c in the interval I,

$$A_f(a;c) = A_f(a;b) + A_f(b;c).$$

2. Let I denote an interval where f is defined and piecewise continuous. Explain why the following properties of the area function are true.

   (a) For any real numbers a and b in the interval I, $A_f(a;b) = -A_f(b;a)$.

   (b) For any real numbers a, b and c in the interval I,

   $$A_f(a;b) = A_f(c;b) - A_f(c;a).$$

   Deduce therefore that

   $$\int_a^b f(t)\ dt = A_f(c;b) - A_f(c;a),$$

   where c is any point in the interval

3. Let $f(t) = |t|$ for all $t \in \mathbb{R}$.

   (a) Compute the area function $A_f(0;x)$ for all $x \in \mathbb{R}$ and sketch the graph of $y = A_f(0;x)$.

   (b) Use the formula obtained in Part (a) and the result from Problem 3(b) to get a formula for computing

   $$\int_a^x |t|\ dt$$

   for any a and x in $\mathbb{R}$.

4. Let I denote an interval where f is defined and piecewise continuous. Explain why the following property of the area function is true.

   $$|A_f(a;x)| \leqslant A_{|f|}(a;x), \qquad \text{for all } a \text{ and } x \text{ in } I \text{ with } a \leqslant x; \tag{3}$$

   that is, the absolute value of the area function of f over $[a,x]$ is at most the area function of the absolute value of f over $[a,x]$.

   Give an example in which strict inequality in (3) holds true.

   Give an example in which equality in (3) holds true.

5. Let I denote an interval where f is defined and piecewise continuous, and let $a \in I$.

   (a) Suppose $f(t) > 0$ for all t in I. Explain why $A_f(a;x)$ increases as x increases over the interval I.

   (b) Suppose $f(t) < 0$ for all t in I. Explain why $A_f(a;x)$ decreases as x increases over the interval I.