

## Assignment #2

Due on Monday, September 16, 2013

Read Sections 2.1 and 2.3 on  $\sigma$ -fields in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Sections 2.2 on *Some Set Algebra* in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Section 1.4 on *Set Theory* in DeGroot and Schervish.

**Background and Definitions**

- A  $\sigma$ -field,  $\mathcal{B}$ , is a collection of subsets of a sample space  $\mathcal{C}$ , referred to as **events**, which satisfy:
  - (1)  $\emptyset \in \mathcal{B}$  ( $\emptyset$  denotes the empty set)
  - (2) If  $E \in \mathcal{B}$ , then its complement,  $E^c$ , is also an element of  $\mathcal{B}$ .
  - (3) If  $(E_1, E_2, E_3 \dots)$  is a sequence of events, then

$$E_1 \cup E_2 \cup E_3 \cup \dots = \bigcup_{k=1}^{\infty} E_k \in \mathcal{B}.$$

- Let  $\mathcal{S}$  denote a collection of subsets of a sample space  $\mathcal{C}$ . The  $\sigma$ -field generated by  $\mathcal{S}$ , denoted by  $\mathcal{B}(\mathcal{S})$ , is the smallest  $\sigma$ -field in  $\mathcal{C}$  which contains  $\mathcal{S}$ .
- $\mathcal{B}_o$  denotes the Borel  $\sigma$ -field of the real line,  $\mathbb{R}$ . This is the  $\sigma$ -field generated by the semi-infinite intervals

$$(-\infty, b], \quad \text{for } b \in \mathbb{R}.$$

Do the following problems

1. Let  $A$ ,  $B$  and  $C$  be subsets of a sample space  $\mathcal{C}$ . Prove the following
  - (a) If  $A \subseteq C$  and  $B \subseteq C$ , then  $A \cup B \subseteq C$ .
  - (b) If  $C \subseteq A$  and  $C \subseteq B$ , then  $C \subseteq A \cap B$ .

2. Let  $\mathcal{C}$  be a sample space and  $\mathcal{B}$  be a  $\sigma$ -field of subsets of  $\mathcal{C}$ . Prove that if  $\{E_1, E_2, E_3, \dots\}$  is a sequence of events in  $\mathcal{B}$ , then

$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{B}.$$

*Hint:* Use De Morgan's Laws.

3. Let  $\mathcal{C}$  be a sample space and  $\mathcal{B}$  be a  $\sigma$ -field of subsets of  $\mathcal{C}$ . For fixed  $B \in \mathcal{B}$  define the collection of subsets

$$\mathcal{B}_B = \{D \subset \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B}\}.$$

Show that  $\mathcal{B}_B$  is a  $\sigma$ -field.

*Note:* In this case, the complement of  $D \in \mathcal{B}_B$  has to be understood as  $B \setminus D$ ; that is, the complement relative to  $B$ . The  $\sigma$ -field  $\mathcal{B}_B$  is the  $\sigma$ -field  $\mathcal{B}$  restricted to  $B$ , or *conditioned on  $B$* .

4. Let  $\mathcal{S}$  denote the collection of all bounded, open intervals  $(a, b)$ , where  $a$  and  $b$  are real numbers with  $a < b$ . Show that

$$\mathcal{B}(\mathcal{S}) = \mathcal{B}_o;$$

that is, the  $\sigma$ -field generated by bounded open intervals is the Borel  $\sigma$ -field.

*Hints:*

- We have already seen in the lecture that  $\mathcal{B}_o$  contains all bounded open intervals.
  - Observe also that the semi-infinite open interval  $(b, \infty)$  can be expressed as the union of the sequence of bounded intervals  $(b, k)$ , for  $k = 1, 2, 3, \dots$
5. Show that for every real number  $a$ , the singleton  $\{a\}$  is in the Borel  $\sigma$ -field  $\mathcal{B}_o$ .
- Hint:* Express  $\{a\}$  as an intersection of a sequence of open intervals.