Review Problems for Exam 2

1. Let $f_X(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } 1 < x < \infty; \\ 0 & \text{if } x \leq 1, \end{cases}$ be the pdf of a random variable X. If E_1 denote the interval $(1,2)$ and E_2 the interval $(4,5)$, compute $\Pr(E_1)$, $\Pr(E_2)$, $\Pr(E_1 \cup E_2)$ and $\Pr(E_1 \cap E_2)$.

2. Let X have pdf $f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}$

   Compute the probability that X is at least $3/4$, given that X is at least $1/2$.

3. Divide a segment at random into two parts. Find the probability that the largest segment is at least three times the shorter.

4. Let X have pdf $f_X(x) = \begin{cases} x^2/9, & \text{if } 0 < x < 3; \\ 0, & \text{elsewhere.} \end{cases}$

   Find the pdf of $Y = X^3$.

5. Let X and Y be independent Normal$(0,1)$ random variables. Put $Z = \dfrac{Y}{X}$. Compute the distribution functions $F_Z(z)$ and $f_Z(z)$.

6. A random point (X,Y) is distributed uniformly on the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

   (a) Give the joint pdf for X and Y.

   (b) Compute the following probabilities: (i) $P(X^2+Y^2 < 1)$, (ii) $P(2X-Y > 0)$, (iii) $P(|X+Y| < 2)$.

7. Prove that if the joint cdf of X and Y satisfies

$$F_{X,Y}(x,y) = F_X(x)F_Y(y),$$

   then for any pair of intervals (a,b) and (c,d),

$$P(a < X \leq b, c < Y \leq d) = P(a < X \leq b)P(c < Y \leq d).$$

$X\backslash Y$	2	3	4
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0

Table 1: Joint Probability Distribution for X and Y, $p_{(X,Y)}$

8. The random pair (X, Y) has the joint distribution shown in Table 1.
   (a) Show that X and Y are not independent.
   (b) Give a probability table for random variables U and V that have the same marginal distributions as X and Y, respectively, but are independent.

9. Let X denote the number of trials needed to obtain the first head, and let Y be the number of trials needed to get two heads in repeated tosses of a fair coin. Are A and Y independent random variables?

10. Let $X \sim \text{Normal}(0, 1)$ and put $Y = X^2$. Find the pdf for Y.

11. Let X and Y be independent Normal$(0, 1)$ random variables.
    Compute $\Pr(X^2 + Y^2 < 1)$.

12. Suppose that X and Y are independent random variables such that $X \sim \text{Uniform}(0, 1)$ and $Y \sim \text{Exponential}(1)$.
    (a) Let $Z = X + Y$. Find F_Z and f_Z.
    (b) Let $U = Y/X$. Find F_U and f_U.

13. Let $X \sim \text{Exponential}(1)$, and define Y to be the integer part of $X + 1$; that is, $Y = i+1$ if and only if $i \leq X < i+1$, for $i = 0, 1, 2, \ldots$ Find the pmf of Y, and deduce that $Y \sim \text{Geometric}(p)$ for some $0 < p < 1$. What is the value of p?

14. Let $X_1, X_2, X_3, \ldots, X_n$ be independent identically distributed Bernoulli random variables with parameter p, with $0 < p < 1$. Define
    $$Y = X_1 + X_2 + \cdots + X_n.$$
    Use moment generating functions to determine the distribution of Y.