Solutions to Review Problems for Exam 2

1. Let $f_X(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } 1 < x < \infty; \\ \\ 0 & \text{if } x \leq 1, \end{cases}$ be the pdf of a random variable X. If E_1 denote the interval $(1, 2)$ and E_2 the interval $(4, 5)$, compute $\Pr(E_1)$, $\Pr(E_2)$, $\Pr(E_1 \cup E_2)$ and $\Pr(E_1 \cap E_2)$.

   ***Solution**:* Compute

$$\Pr(E_1) = \int_1^2 \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_1^2 = \frac{1}{2},$$

$$\Pr(E_2) = \int_4^5 \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_4^5 = \frac{1}{20},$$

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) = \frac{11}{20},$$

   since E_1 and E_2 are mutually exclusive, and

$$\Pr(E_1 \cap E_2) = 0,$$

   since E_1 and E_2 are mutually exclusive. □

2. Let X have pdf $f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}$

   Compute the probability that X is at least 3/4, given that X is at least 1/2.

   ***Solution**:* We are asked to compute

$$\Pr(X \geqslant 3/4 \mid X \geqslant 1/2) = \frac{\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)]}{\Pr(X \geqslant 1/2)}, \tag{1}$$

   where

$$\begin{aligned} \Pr(X \geqslant 1/2) &= \int_{1/2}^1 2x\, dx \\ &= x^2\Big|_{1/2}^1 \\ &= 1 - \frac{1}{4}, \end{aligned}$$

so that

$$\Pr(X \geqslant 1/2) = \frac{3}{4}; \tag{2}$$

and

$$\begin{aligned} \Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)] &= \Pr(X \geqslant 3/4) \\ &= \int_{3/4}^{1} 2x\ dx \\ &= x^2\Big|_{3/4}^{1} \\ &= 1 - \frac{9}{16}, \end{aligned}$$

so that

$$\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)] = \frac{7}{16}. \tag{3}$$

Substituting (3) and (2) into (1) then yields

$$\Pr(X \geqslant 3/4 \mid X \geqslant 1/2) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{12}.$$

□

3. Divide a segment at random into two parts. Find the probability that the largest segment is at least three times the shorter.

   ***Solution**:* Assume the segment is the interval $(0,1)$ and let $X \sim \text{Uniform}(0,1)$. Then X models a random point in $(0,1)$. Let A denote the event that the longer of X or $1-X$ is at least three times the length of the shorter of the two.

   We have two possibilities: Either $X \leqslant 1 - X$ or $X > 1 - X$; or, equivalently, $X \leqslant \frac{1}{2}$ or $X > \frac{1}{2}$. Define the events $E_1 = \left(X \leqslant \frac{1}{2}\right)$ and $E_2 = \left(X > \frac{1}{2}\right)$. Observe that E_1 and E_2 are mutually exclusive. Observe also that if E_1 occurs, then A corresponds to $1 - X \geqslant 3X$, or $X \leqslant 1/4$; and, if E_2 occurs, then A corresponds to $X \geqslant 3(1-X)$, or $X \geqslant 3/4$. Thus, by the Law of Total Probability,

$$\begin{aligned} \Pr(A) &= \Pr(A \cap E_1) + \Pr(A \cap E_2) \\ &= \Pr\left(X \leqslant \frac{1}{2}, X \leqslant \frac{1}{4}\right) + \Pr\left(X > \frac{1}{2}, X \geqslant \frac{3}{4}\right), \end{aligned}$$

so that

$$\Pr(A) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Thus, the probability that the largest segment is at least three times the shorter is 1/2. □

4. Let X have pdf $f_X(x) = \begin{cases} x^2/9, & \text{if } 0 < x < 3; \\ 0, & \text{elsewhere.} \end{cases}$

   Find the pdf of $Y = X^3$.

   ***Solution**:* First, compute the cdf of Y

$$F_Y(y) = \Pr(Y \leqslant y). \tag{4}$$

Observe that, since $Y = X^3$ and the possible values of X range from 0 to 3, the values of Y will range from 0 to 27. Thus, in the calculations that follow, we will assume that $0 < y < 27$.

From (4) we get that

$$\begin{aligned} F_Y(y) &= \Pr(X^3 \leqslant y) \\ &= \Pr(X \leqslant y^{1/3}) \\ &= F_X(y^{1/3}) \end{aligned}$$

Thus, for $0 < y < 27$, we have that

$$f_Y(y) = f_X(y^{1/3}) \cdot \frac{1}{3} y^{-3/2}, \tag{5}$$

where we have applied the Chain Rule. It follows from (5) and the definition of f_X that

$$f_Y(y) = \frac{1}{9}\left[y^{1/3}\right]^2 \cdot \frac{1}{3} y^{-3/2} = \frac{1}{27}, \qquad \text{for } 0 < y < 27. \tag{6}$$

Combining (6) and the definition of f_X we obtain the pdf for Y:

$$f_Y(y) = \begin{cases} \dfrac{1}{27}, & \text{for } 0 < y < 27; \\ 0 & \text{elsewhere;} \end{cases}$$

in other words $Y \sim \text{Uniform}(0, 27)$. □

5. Let X and Y be independent Normal(0, 1) random variables. Put $Z = \dfrac{Y}{X}$. Compute the distribution functions $F_Z(z)$ and $f_Z(z)$.

Solution: Since $X, Y \sim$ Normal(0, 1), their pdfs are given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \qquad \text{for } x \in \mathbb{R},$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}, \qquad \text{for } y \in \mathbb{R},$$

respectively. The joint pdf of (X, Y) is then

$$f_{(X,Y)}(x, y) = \frac{1}{2\pi}\, e^{-(x^2+y^2)/2}, \qquad \text{for } (x, y) \in \mathbb{R}^2. \tag{7}$$

We compute the cdf of Z,

$$F_Z(z) = \Pr(Z \leqslant z) = \Pr\left(\frac{y}{x} \leqslant z\right),$$

or

$$F_Z(z) = \iint_{\frac{y}{x}\leqslant z} f_{(X,Y)}(x, y)\; dxdy, \tag{8}$$

where the integrand in (8) is given in (7) and the integration is done over the region

$$R = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{y}{x} \leqslant z\right\}.$$

Make the change variables

$$\begin{aligned} u &= x \\ v &= \frac{y}{x}, \end{aligned}$$

so that

$$\begin{aligned} x &= u \\ y &= uv, \end{aligned} \tag{9}$$

in the integral in (8) to obtain

$$F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty} f_{(X,Y)}(u, uv) \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\; dudv, \tag{10}$$

where

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} 1 & 0 \\ v & u \end{pmatrix} = u, \tag{11}$$

is the Jacobian determinant of the transformation in (9). It then follows from (10) and (11) that

$$F_Z(z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_{(X,Y)}(u, uv)\,|u|\; dudv, \tag{12}$$

Differentiating with respect to z and using the definition of the joint pdf of (X, Y) in (7) we obtain from (12) that

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |u|\; e^{-(1+z^2)u^2/2}\; du, \tag{13}$$

where we have also used the Fundamental Theorem of Calculus.

Since the integrand in (13) is an even function of u, we can rewrite the expression for f_Z in (13) as

$$f_Z(z) = \frac{1}{\pi} \int_{0}^{\infty} u\; e^{-(1+z^2)u^2/2}\; du. \tag{14}$$

Integrating the right–hand side of equation in (14) we obtain

$$f_Z(z) = \frac{1}{\pi} \cdot \frac{1}{1+z^2}, \qquad \text{for } z \in \mathbb{R}. \tag{15}$$

The cdf of Z is then obtained by integrating (15) to get

$$F_Z(z) = \int_{-\infty}^{z} f_Z(z)\; dz = \frac{1}{2} + \frac{1}{\pi}\arctan(z), \qquad \text{for } z \in \mathbb{R}.$$

□

6. A random point (X, Y) is distributed uniformly on the square with vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$.

   (a) Give the joint pdf for X and Y.

   (b) Compute the following probabilities:

      (i) $\Pr(X^2 + Y^2 < 1)$,

      (ii) $\Pr(2X - Y > 0)$,

      (iii) $\Pr(|X + Y| < 2)$.

***Solution**:* The square is pictured in Figure 1 and has area 4.

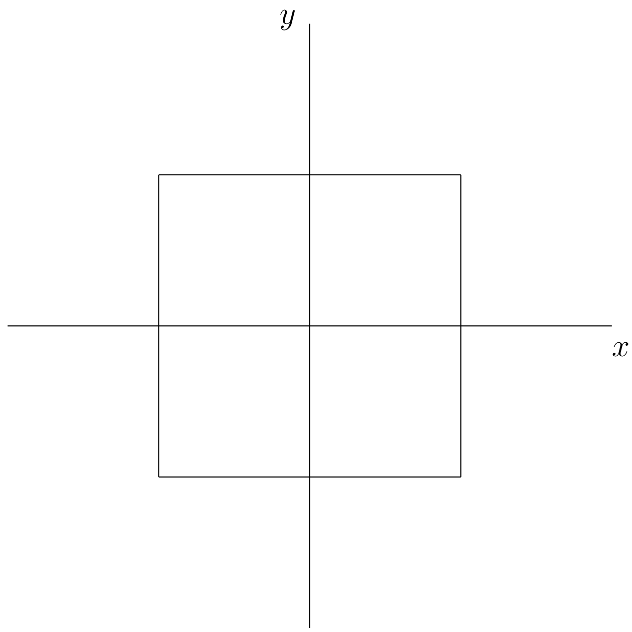

Figure 1: Sketch of square in Problem 6

(a) Consequently, the joint pdf of (X,Y) is given by

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{4}, & \text{for } -1 < x < 1, -1 < y < 1; \\ \\ 0 & \text{elsewhere.} \end{cases} \tag{16}$$

(b) Denoting the square in Figure 1 by R, it follows from (16) that, for any subset A of $\mathbb{R}^2$,

$$\Pr[(x,y) \in A] = \iint_A f_{(X,Y)}(x,y)\ dxdy = \frac{1}{4} \cdot \text{area}(A \cap R); \tag{17}$$

that is, $\Pr[(x,y) \in A]$ is one–fourth the area of the portion of A in R.

We will use the formula in (17) to compute each of the probabilities in (i), (ii) and (iii).

(i) In this case, A is the circle of radius 1 around the origin in $\mathbb{R}^2$ and pictured in Figure 2.

Note that the circle A in Figure 2 is entirely contained in the square R so that, by the formula in (17),

$$\Pr(X^2 + Y^2 < 1) = \frac{\text{area}(A)}{4} = \frac{\pi}{4}.$$

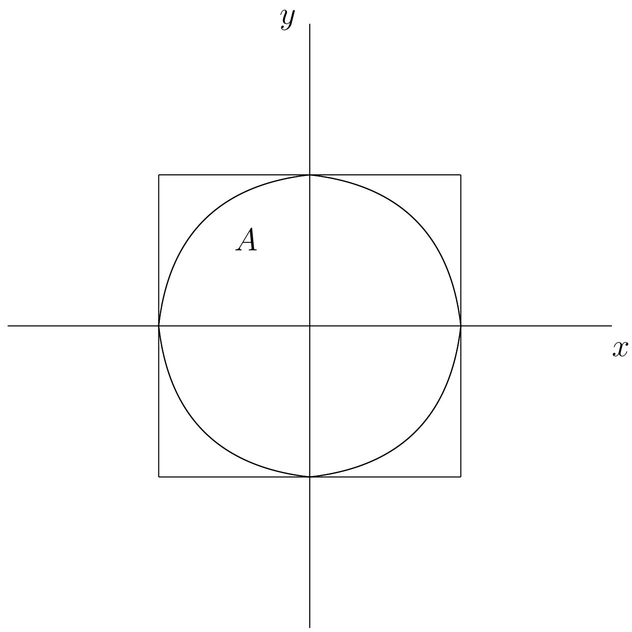

Figure 2: Sketch of A in Problem 6(i)

(ii) The set A in this case is pictured in Figure 3 on page 8. Thus, in this case, $A \cap R$ is a trapezoid of area $2 \cdot \dfrac{\frac{1}{2} + \frac{3}{2}}{2} = 2$, so that, by the formula in (17),

$$\Pr(2X - Y > 0) = \frac{1}{4} \cdot \text{area}(A \cap R) = \frac{1}{2}.$$

(iii) In this case, A is the region in the xy–plane between the lines $x+y=2$ and $x+y=-2$ (see Figure 4 on page 9). Thus, $A \cap R$ is R so that, by the formula in (17),

$$\Pr(|X + Y| < 2) = \frac{\text{area}(R)}{4} = 1.$$

□

7. Prove that if the joint cdf of X and Y satisfies

$$F_{(X,Y)}(x, y) = F_X(x) F_Y(y),$$

then for any pair of intervals (a, b) and (c, d),

$$\Pr(a < X \leq b, c < Y \leq d) = \Pr(a < X \leqslant b) \Pr(c < Y \leqslant d).$$

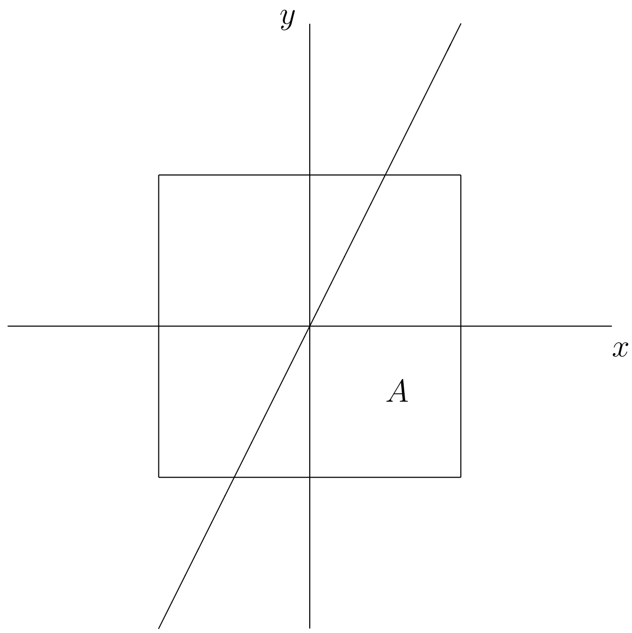

Figure 3: Sketch of A in Problem 6(ii)

Solution: First show that

$$\Pr(a < X \leq b, c < Y \leq d) = F_{(X,Y)}(b,d) - F_{(X,Y)}(b,c) - F_{(X,Y)}(a,d) + F_{(X,Y)}(a,c)$$

(see Problem 1 in Assignment #15). Then,

$$\begin{aligned}
\Pr(a < X \leq b, c < Y \leq d) &= F_X(b)F_Y(d) - F_X(b)F_Y(c) \\
&\quad - F_X(a)F_Y(d) + F_X(a)F_Y(c) \\
&= (F_X(b) - F_X(a))F_Y(d) \\
&\quad - (F_X(b) - F_X(a))F_Y(c) \\
&= (F_X(b) - F_X(a))(F_Y(d) - F_Y(c)) \\
&= \Pr(a < X \leqslant b)\Pr(c < Y \leqslant d),
\end{aligned}$$

which was to be shown. □

8. The random pair (X, Y) has the joint distribution shown in Table 1 on page 9.

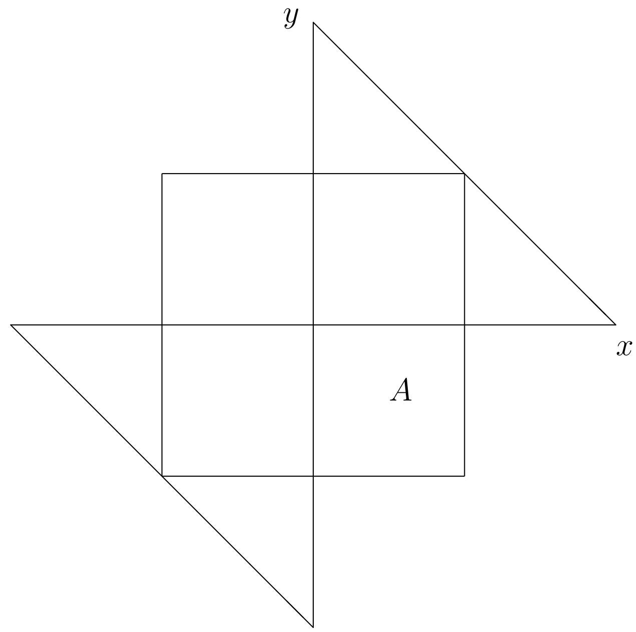

Figure 4: Sketch of A in Problem 6(iii)

$X\backslash Y$	2	3	4
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0

Table 1: Joint Probability Distribution for (X, Y), $p_{(X,Y)}$, in Problem 8

(a) Show that X and Y are not independent.

***Solution**:* Table 2 shows the marginal distributions of X and Y on the margins on page 10.

Observe from Table 2 that

$$p_{(X,Y)}(1, 4) = 0,$$

while

$$p_X(1) = \frac{1}{4} \quad \text{and} \quad p_Y(4) = \frac{1}{3}.$$

Thus,

$$p_X(1) \cdot p_Y(4) = \frac{1}{12};$$

$X\backslash Y$	2	3	4	p_X
1	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{4}$
2	$\frac{1}{6}$	0	$\frac{1}{3}$	$\frac{1}{2}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{4}$
p_Y	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Table 2: Joint pdf for X and Y and marginal distributions p_X and p_Y

so that

$$p_{(X,Y)}(1,4) \neq p_X(1) \cdot p_Y(4),$$

and, therefore, X and Y are not independent. □

(b) Give a probability table for random variables U and V that have the same marginal distributions as X and Y, respectively, but are independent.

Solution: Table 3 on page 10 shows the joint pmf of (U, V) and the marginal distributions, p_U and p_V. □

$U\backslash V$	2	3	4	p_U
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
3	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
p_V	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Table 3: Joint pdf for U and V and their marginal distributions.

9. Let X denote the number of trials needed to obtain the first head, and let Y be the number of trials needed to get two heads in repeated tosses of a fair coin. Are X and Y independent random variables?

Solution: X has a geometric distribution with parameter $p = \dfrac{1}{2}$, so that

$$p_X(k) = \frac{1}{2^k}, \quad \text{for } k = 1, 2, 3, \ldots \tag{18}$$

On the other hand,

$$\Pr[Y = 2] = \frac{1}{4}, \tag{19}$$

since, in two repeated tosses of a coin, the events are HH, HT, TH and TT, and these events are equally likely.

Next, consider the joint event $(X = 2, Y = 2)$. Note that

$$(X = 2, Y = 2) = [X = 2] \cap [Y = 2] = \emptyset,$$

since $[X = 2]$ corresponds to the event TH, while $[Y = 2]$ to the event HH. Thus,

$$\Pr(X = 2, Y = 2) = 0,$$

while

$$p_X(2) \cdot p_Y(2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16},$$

by (18) and (19). Thus,

$$p_{(X,Y)}(2, 2) \neq p_X(2) \cdot p_X(2).$$

Hence, X and Y are not independent. □

10. Let $X \sim \text{Normal}(0, 1)$ and put $Y = X^2$. Find the pdf for Y.

***Solution**:* The pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for } -\infty < x < \infty.$$

We compute the pdf for Y by first determining its cdf:

$$\begin{aligned}
\Pr(Y \leqslant y) &= P(X^2 \leqslant y) \quad \text{for } y \geqslant 0 \\
&= \Pr(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) \\
&= \Pr(-\sqrt{y} < X \leqslant \sqrt{y}), \quad \text{since X is continuous.}
\end{aligned}$$

Thus,

$$\begin{aligned}
\Pr(Y \leqslant y) &= \Pr(X \leqslant \sqrt{y}) - \Pr(X \leqslant -\sqrt{y}) \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad \text{for } y > 0.
\end{aligned}$$

We then have that the cdf of Y is

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad \text{for } y > 0,$$

from which we get, after differentiation with respect to y,

$$\begin{aligned}
f_Y(y) &= F'_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + F'_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \\
&= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}} \left\{ \frac{1}{\sqrt{2\pi}}\ e^{-y/2} + \frac{1}{\sqrt{2\pi}}\ e^{-y/2} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y}}\ e^{-y/2},
\end{aligned}$$

for $y > 0$, where we have applied the Chain Rule. Hence,

$$f_{_Y}(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \cdot \dfrac{1}{\sqrt{y}}\, e^{-y/2}, & \text{for } y > 0; \\ \\ 0 & \text{for } y \leqslant 0. \end{cases}$$

□

11. Let X and Y be independent Normal(0, 1) random variables. Compute

$$P(X^2 + Y^2 < 1).$$

Solution: Since $X, Y \sim \text{Normal}(0, 1)$, their pdfs are given by

$$f_{_X}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \qquad \text{for } x \in \mathbb{R},$$

and

$$f_{_Y}(y) = \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}, \qquad \text{for } y \in \mathbb{R},$$

respectively. The joint pdf of (X, Y) is then

$$f_{_{(X,Y)}}(x, y) = \frac{1}{2\pi}\, e^{-(x^2+y^2)/2}, \qquad \text{for } (x, y) \in \mathbb{R}^2. \tag{20}$$

Thus,

$$P(X^2 + Y^2 < 1) = \iint_{x^2+y^2<1} f_{_{(X,Y)}}(x, y)\, dxdy, \tag{21}$$

where the integrand is given in (20) and the integral in (21) is evaluated over the disc of radius 1 centered around the origin in $\mathbb{R}^2$.

We evaluate the integral in (21) by changing to polar coordinates to get

$$\begin{aligned} P(X^2 + Y^2 < 1) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 e^{-r^2/2} r dr d\theta \\ &= \int_0^1 e^{-r^2/2} r dr \\ &= \left[-e^{-r^2/2} \right]_0^1 \\ &= 1 - e^{-1/2}, \end{aligned}$$

or $\Pr(X^2 + Y^2 < 1) = 1 - \dfrac{1}{\sqrt{e}}$. □

12. Suppose that X and Y are independent random variables such that $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Exponential}(1)$.

    (a) Let $Z = X + Y$. Find F_Z and f_Z.

    ***Solution**:* Since $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Exponential}(1)$, their pdfs are given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0; \\ 0 & \text{if } y \leqslant 0, \end{cases}$$

respectively. The joint pdf of (X, Y) is then

$$f_{(X,Y)}(x, y) = \begin{cases} e^{-y} & \text{if } 0 < x < 1,\ y > 0; \\ 0 & \text{elsewhere.} \end{cases} \tag{22}$$

We compute the cdf of Z,

$$F_Z(z) = \Pr(X \leqslant u) = \Pr\left(X + Y \leqslant z\right),$$

or

$$F_U(u) = \iint_{x+y\leqslant z} f_{(X,Y)}(x, y)\ dxdy, \tag{23}$$

where the integrand in (23) is given in (22) and the integration is done over the region

$$R = \left\{(x, y) \in \mathbb{R}^2 \mid x + y \leqslant z\right\}.$$

Make the change variables

$$\begin{aligned} u &= x \\ v &= x + y, \end{aligned}$$

so that

$$\begin{aligned} x &= u \\ y &= v - u, \end{aligned} \tag{24}$$

in the integral in (23) to obtain

$$F_Z(z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_{(X,Y)}(u, v - u) \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\ dudv, \tag{25}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1, \tag{26}$$

is the Jacobian determinant of the transformation in (24). It then follows from (25) and (26) that

$$F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty} f_{(X,Y)}(u, v-u)\ dudv. \tag{27}$$

Differentiating with respect to z and using the definition of the joint pdf of (X,Y) in (22) we obtain from (27) that

$$f_Z(z) = \int_{-\infty}^{\infty} f_{(X,Y)}(u, z-u)\ du. \tag{28}$$

where we have also used the Fundamental Theorem of Calculus.
Next, use the definition of $f_{(X,Y)}$ in (22) to rewrite (28) as

$$f_Z(z) = \int_0^1 f_{(X,Y)}(u, z-u)\ du, \quad \text{for } z > 0, \tag{29}$$

We consider two cases, (i) $0 < z \leqslant 1$, and (ii) $z > 1$.

(i) For $0 < z \leqslant 1$, use (22) to obtain from (29) that

$$\begin{aligned} f_Z(z) &= \int_0^z e^{u-z}\ du \\ &= e^{-z}\int_0^z e^u\ du \\ &= 1 - e^{-z}, \end{aligned}$$

so that

$$f_Z(z) = 1 - e^{-z}, \quad \text{for } 0 < z \leqslant 1. \tag{30}$$

(ii) For $z > 1$, use (22) to obtain from (29) that

$$\begin{aligned} f_Z(z) &= \int_0^1 e^{u-z}\ du \\ &= e^{-z}\int_0^1 e^u\ du \\ &= (e-1)e^{-z}, \end{aligned}$$

so that

$$f_Z(z) \quad = \quad (e-1)e^{-z}, \qquad \text{for } z > 1. \tag{31}$$

Combining (30) and (31) we obtain the cdf

$$f_Z(z) = \begin{cases} 0 & \text{for } z \leqslant 0; \\ 1 - e^{-z}, & \text{for } 0 < z \leqslant 1; \\ (e-1)e^{-z}, & \text{for } z > 1. \end{cases} \tag{32}$$

Finally, integrating (32) yields the cdf

$$F_Z(z) = \begin{cases} 0 & \text{for } z \leqslant 0; \\ z + e^{-z} - 1, & \text{for } 0 < z \leqslant 1; \\ e^{-1} + (e-1)(e^{-1} - e^{-z}), & \text{for } z > 1. \end{cases}$$

□

(b) Let $U = Y/X$. Find F_U and f_U.

***Solution**:* Since $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Exponential}(1)$, their pdfs are given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0; \\ 0 & \text{if } y \leqslant 0, \end{cases}$$

respectively. The joint pdf of (X, Y) is then

$$f_{(X,Y)}(x,y) = \begin{cases} e^{-y} & \text{if } 0 < x < 1, \ y > 0; \\ 0 & \text{elsewhere.} \end{cases} \tag{33}$$

We compute the cdf of U,

$$F_U(u) = \Pr(U \leqslant u) = \Pr\left(\frac{Y}{X} \leqslant u\right),$$

or

$$F_U(u) = \iint_{\frac{y}{x} \leqslant u} f_{(X,Y)}(x,y)\, dxdy, \tag{34}$$

where the integrand in (34) is given in (33) and the integration is done over the region

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{y}{x} \leqslant u\right\}.$$

Make the change variables

$$\begin{aligned} w &= x \\ v &= \frac{y}{x}, \end{aligned}$$

so that

$$\begin{aligned} x &= w \\ y &= wv, \end{aligned} \tag{35}$$

in the integral in (34) to obtain

$$F_U(u) = \int_{-\infty}^{u} \int_{-\infty}^{\infty} f_{(X,Y)}(w, wv) \left| \frac{\partial(x,y)}{\partial(w,v)} \right| \, dwdv, \tag{36}$$

where

$$\frac{\partial(x,y)}{\partial(w,v)} = \det \begin{pmatrix} 1 & 0 \\ v & w \end{pmatrix} = w, \tag{37}$$

is the Jacobian determinant of the transformation in (35). It then follows from (36) and (37) that

$$F_U(u) = \int_{-\infty}^{u} \int_{-\infty}^{\infty} f_{(X,Y)}(w, wv) \, |w| \, dwdv. \tag{38}$$

Differentiating with respect to u and using the definition of the joint pdf of (X, Y) in (33) we obtain from (38) that

$$f_U(u) = \int_{-\infty}^{\infty} f_{(X,Y)}(w, wu)|w| \, dw. \tag{39}$$

where we have also used the Fundamental Theorem of Calculus.

Next, use the definition of $f_{(X,Y)}$ in (33) to rewrite (39) as

$$f_U(u) = \int_0^1 e^{-uw} w \, dw, \quad \text{for } u > 1, \tag{40}$$

We evaluate the integral in (40) by integration by parts to get

$$\begin{aligned} f_U(u) &= \left[-\frac{w}{u} \, e^{-uw} - \frac{1}{u^2} \, e^{-uw} \right]_0^1 \\ &= \frac{1}{u^2} - \frac{1}{u} \, e^{-u} - \frac{1}{u^2} \, e^{-u}, \quad \text{for } u > 0. \end{aligned} \tag{41}$$

In order to compute the cdf, F_U, we can integrate (34) in Cartesian coordinates to get

$$\begin{aligned} F_U(u) &= \int_0^1 \int_0^{ux} e^{-y}\, dydx \\ &= \int_0^1 [1 - e^{-ux}]\, dx \\ &= 1 + \frac{1}{u}[e^{-u} - 1], \end{aligned}$$

so that

$$F_U(u) = \begin{cases} 1 + \dfrac{1}{u}[e^{-u} - 1], & \text{for } u > 0; \\ \\ 0 & \text{for } u \leqslant 0. \end{cases} \tag{42}$$

Note that differentiating $F_U(u)$ in (42) with respect to u, for $u > 0$, leads to (41). We then have that

$$f_U(u) = \begin{cases} \dfrac{1}{u^2}(1 - e^{-u}) - \dfrac{1}{u}\, e^{-u}, & \text{for } u > 0; \\ \\ 0 & \text{for } u \leqslant 0. \end{cases}$$

□

13. Let $X \sim \text{Exponential}(1)$, and define Y to be the integer part of $X + 1$; that is, $Y = i + 1$ if and only if $i \leqslant X < i + 1$, for $i = 0, 1, 2, \ldots$ Find the pmf of Y, and deduce that $Y \sim \text{Geometric}(p)$ for some $0 < p < 1$. What is the value of p?

    ***Solution**:* Compute

    $$\Pr[Y = i + 1] = \Pr[i \leqslant X < i + 1] = \Pr[i < X \leqslant i + 1],$$

    since X is continuous; so that

    $$\Pr[Y = i + 1] = \int_i^{i+1} f_X(x)\, dx, \tag{43}$$

    where

    $$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0; \\ 0 & \text{if } x \leqslant 0, \end{cases} \tag{44}$$

    since $X \sim \text{Exponential}(1)$.

Evaluating the integral in (43), for $i \geqslant 0$ and f_X as given in (44), yields

$$\begin{aligned}\Pr[Y = i+1] &= \int_i^{i+1} e^{-x}\, dx \\ &= \left[-e^{-x}\right]_i^{i+1} \\ &= e^{-i} - e^{-i-1},\end{aligned}$$

so that

$$\Pr[Y = i+1] = \left(\frac{1}{e}\right)^i \left(1 - \frac{1}{e}\right) \tag{45}$$

It follows from (45) that $Y \sim \text{Geometric}(p)$ with $p = 1 - \dfrac{1}{e}$. □

14. Let $X_1, X_2, X_3, \ldots, X_n$ be independent identically distributed Bernoulli random variables with parameter p, with $0 < p < 1$. Define

$$Y = X_1 + X_2 + \cdots + X_n.$$

Use moment generating functions to determine the distribution of Y.

***Solution**:* Compute the moment generation function of Y to get

$$\begin{aligned}\psi_Y(t) &= E(e^{tY}) \\ &= E\big(e^{t(X_1+X_2+\cdots+X_n)}\big) \\ &= E\big(e^{tX_1+tX_2+\cdots+tX_n}\big) \\ &= E(e^{tX_1}e^{tX_2}\cdots e^{tX_n}),\end{aligned}$$

so that

$$\psi_Y(t) = E(e^{tX_1})\cdot E(e^{tX_2})\cdots E(e^{tX_n}),$$

since the random variables $X_1, X_2, \ldots, X_n$ are mutually independent.

It then follows that

$$\begin{aligned}\psi_Y(t) &= \psi_{X_1}(t)\cdot\psi_{X_2}(t)\cdots\psi_{X_2}(t) \\ &= (pe^t + 1 - p)\cdot(pe^t + 1 - p)\cdots(pe^t + 1 - p),\end{aligned}$$

since each of the X_i has a Bernoulli(p) distribution.

We then have that

$$\psi_{_Y}(t) = (pe^t + 1 - p)^n, \quad \text{for all } t \in \mathbb{R},$$

which is the moment generating function for a Binomial(n, p) distribution. It then follows from the Uniqueness Theorem for moment generating functions that Y has a Binomial(n, p) distribution. Hence, the pmf for Y is

$$p_{_Y}(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & \text{for } k = 0, 1, 2, \ldots, n; \\ \\ 0, & \text{elsewhere.} \end{cases}$$

□