Solutions to Exam 2

1. In each of the following, X and Y denote independent random variables. In each case, set $Z = X + Y$ and compute the mgf, ψ_Z, of Z; then use ψ_Z to determine the distribution of Z.

   (a) $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$, where m and n are positive integers and $0 < p < 1$.

   Solution: Compute

$$\begin{aligned} \psi_Z(t) &= E(e^{tZ}) \\ &= E(e^{t(X+Y)}) \\ &= E(e^{tX+tY}) \\ &= E(e^{tX} \cdot e^{tY}), \end{aligned}$$

   so that
$$\psi_Z(t) = E(e^{tX}) \cdot E(e^{tY}), \tag{1}$$
   since we are assuming that X and Y are independent.

   It follows from (1) and the definition of the moment generating function that
$$\psi_Z(t) = \psi_X(t) \cdot \psi_Y(t). \tag{2}$$
   For the case in which $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$, we obtain from (2) that

$$\psi_Z(t) = (pe^t + 1 - p)^n \cdot (pe^t + 1 - p)^m, \quad \text{for } t \in \mathbb{R},$$

   so that
$$\psi_Z(t) = (pe^t + 1 - p)^{n+m}, \quad \text{for } t \in \mathbb{R}. \tag{3}$$
   It follows from (3) and the uniqueness theorem for moment generating functions that
$$Z \sim \text{Binomial}(n + m, p).$$
   □

   (b) $X \sim \text{Normal}(\mu, 1/\sqrt{2})$ and $Y \sim \text{Normal}(-\mu, 1/\sqrt{2})$, where μ is a real parameter.

   Solution: We proceed as in part (a).

For the case in which $X \sim \text{Normal}(\mu, 1/\sqrt{2})$ and $Y \sim \text{Normal}(-\mu, 1/\sqrt{2})$, it follows from (2) that

$$\begin{aligned}\psi_Z(t) &= e^{\mu t+\frac{1}{2}\cdot\frac{1}{\sqrt{2}}t^2}\cdot e^{-\mu t+\frac{1}{2}\cdot\frac{1}{\sqrt{2}}t^2} \\ &= e^{\frac{1}{\sqrt{2}}t^2}\end{aligned}$$

so that

$$\psi_Z(t) = e^{\frac{1}{2}\cdot\frac{2}{\sqrt{2}}t^2}, \quad \text{for } t \in \mathbb{R},$$

or

$$\psi_Z(t) = e^{\frac{1}{2}\cdot\sqrt{2}\ t^2}, \quad \text{for } t \in \mathbb{R}, \tag{4}$$

It follows from (4) and the uniqueness theorem for moment generating functions that

$$Z \sim \text{Normal}(0, \sqrt{2}).$$

□

2. The moment generating function of a random variable, X, is given by

$$\psi_X(t) = \frac{1}{1-2t}, \quad \text{for } t < \frac{1}{2}.$$

(a) Compute $E(X)$ and $\text{Var}(X)$.

***Solution**:* It follows from the uniqueness theorem for moment generating functions that X has an Exponential(2) distribution, so that

$$E(X) = 2 \quad \text{and} \quad \text{Var}(X) = 4.$$

□

(b) Give the distribution of X and use it to find a value of m for which

$$\Pr(X \leqslant m) = \frac{1}{2}.$$

***Solution**:* Since X has an Exponential(2) distribution, it follows that its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{1}{2}\, e^{-x/2}, & \text{for } x > 0; \\ \\ 0, & \text{for } x \leqslant 0. \end{cases}$$

Using this pdf we compute

$$\Pr(X \leqslant m) = 1 - e^{-m/2},$$

so that

$$\Pr(X \leqslant m) = \frac{1}{2}$$

implies that

$$e^{-m/2} = \frac{1}{2},$$

from which we get that

$$m = 2\ln 2.$$

□

3. Assume that the joint pdf of a random vector (X, Y) is given by the function

$$f(x,y) = \begin{cases} c(2 - xy^2), & \text{for } 1 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 1; \\ 0, & \text{elsewhere,} \end{cases}$$

where c is a positive constant.

(a) Determine the value of c.
***Solution**:* Evaluate the integral

$$\begin{aligned} \iint_{\mathbb{R}^2} f(x,y)\ dxdy &= c\int_1^2 \int_0^1 (2 - xy^2)\ dydx \\ &= c\int_1^2 \left[2y - \frac{1}{3}xy^3\right]_0^1\ dx \\ &= c\int_1^2 \left(2 - \frac{1}{3}x\right)\ dx \\ &= c\left[2x - \frac{1}{6}x^2\right]_1^2 \\ &= c\left(\frac{10}{3} - \frac{11}{6}\right), \end{aligned}$$

so that

$$\iint_{\mathbb{R}^2} f(x,y)\ dxdy = c \cdot \frac{3}{2}.$$

Thus, since we are given that f is a pdf, $c = \dfrac{2}{3}$. □

(b) Determine the marginal distribution, f_X, and compute $E(X)$.
***Solution**:* Compute, for $1 < x < 2$,

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\ dy \\
&= \frac{2}{3}\int_0^1 \left(2 - xy^2\right) dy \\
&= \frac{2}{3}\left[2y - \frac{1}{3}xy^3\right]_0^1
\end{aligned}
$$

so that

$$
f_X(x) = \begin{cases} \dfrac{2}{3}\left(2 - \dfrac{x}{3}\right), & \text{for } 1 < x < 2; \\ \\ 0, & \text{elsewhere.} \end{cases}
$$

To find the expected value of X, compute

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X(x)\ dx \\
&= \int_1^2 x \cdot \frac{2}{3}\left(2 - \frac{x}{3}\right)\ dx \\
&= \frac{2}{3}\int_1^2 \left(2x - \frac{x^2}{3}\right)\ dx \\
&= \frac{2}{3}\left[x^2 - \frac{x^3}{9}\right]_1^2 \\
&= \frac{2}{3}\left[4 - \frac{8}{9} - \left(1 - \frac{1}{9}\right)\right],
\end{aligned}
$$

so that $E(X) = \dfrac{40}{27}$. □

4. Let X denote the time a patient spends at a waiting room of a doctor's office waiting to be seen by a physician, and Y the time the physician actually spends with the patient. Assume that X and Y are independent random variables with

$X \sim$ Exponential(40) and $Y \sim$ Exponetial(20), where X and Y are measured in minutes.

(a) On average, how long will a patient spend at the waiting room, and how long does the patient spends being seen by a doctor?

> ***Answer:*** $E(X) = 40$ and $E(Y) = 20$. Thus, on average, a patient spends 40 minutes in the waiting room, and 20 minutes being seen by a doctor. □

(b) What is the expected value of the time a patient will spend at the doctor's office? Explain your reasoning.

> ***Answer:*** $E(X+Y) = E(X) + E(Y) = 40 + 20 = 60$. Thus, on average, a patient spends 60 minutes the doctor's office. □

(c) Give the joint distribution of (X, Y).

Solution: Since X and Y are independent,

$$f_{(X,Y)}(x,y) = f_X(x) \cdot f_Y(y),$$

where

$$f_X(x) = \begin{cases} \dfrac{1}{40}\, e^{-x/40}, & \text{for } x > 0; \\[2ex] 0, & \text{for } x \leqslant 0, \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{1}{20}\, e^{-y/20}, & \text{for } y > 0; \\[2ex] 0, & \text{for } y \leqslant 0. \end{cases}$$

It then follows that

$$f_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{800}\, e^{-x/40} e^{-y/20}, & \text{for } x > 0 \text{ and } y > 0; \\[2ex] 0, & \text{elsewhere.} \end{cases}$$

□

(d) Set up (but DO NOT EVALUATE) the iterated double integral that yields the probability that a patient will spend less than an hour at a doctor's office.

Solution: We want

$$\Pr(X + Y < 60) = \iint_A f_{(X,Y)}(x, y)\, dxdy,$$

where $A = \{(x, y) \in \mathbb{R}^2 \mid x + y < 60\}$.
Using the definition of the joint pdf for (X, Y) found in the previous part we get

$$\Pr(X + Y < 60) = \int_0^{60} \int_0^{60-x} \frac{1}{800}\, e^{-x/40} e^{-y/20}\, dy\, dx.$$

□