Review Problems for Final Exam

1. Three cards are in a bag. One card is red on both sides. Another card is white on both sides. The third card in red on one side and white on the other side. A card is picked at random and placed on a table. Compute the probability that if a given color is shown on top, the color on the other side is the same as that of the top.

2. An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and a number b of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Determine the value of b.

3. A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

4. A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants). What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

5. Suppose that $0 < \rho < 1$ and let $p(n) = \rho^n(1-\rho)$ for $n = 0, 1, 2, 3, \ldots$

   (a) Verify that p is the probability mass function (pmf) for a random variable.

   (b) Let X denote a discrete random variable with pmf p. Compute $\Pr(X > 1)$.

6. In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying assumption that for all integers $n \geqslant 0$, $p_{n+1} = \frac{1}{5}p_n$, where p_n represents the probability that the policyholder files n claims during the period.

   (a) Find the value of p_o for which p_n defines the probability mass function for the number, N, of claims filed during that period.

   (b) What is the probability that a policyholder files more than one claim during the period?

7. The time, T, that a manufacturing system is out of operation has cumulative distribution function

$$F_{_T}(t) = \begin{cases} 1 - \left(\dfrac{2}{t}\right)^2, & \text{for } t > 2; \\ \\ 0, & \text{elsewhere.} \end{cases}$$

The resulting cost to the company is proportional to $Y = T^2$. Determine the probability density function for Y.

8. Let $N(t)$ denote the number of mutations in a bacterial colony that occur during the interval $[0, t)$. Assume that $N(t) \sim \text{Poisson}(\lambda t)$ where $\lambda > 0$ is a positive parameter.

   (a) Give an interpretation for λ.

   (b) Let T_1 denote the time that the first mutation occurs. Find the distribution of T_1.

9. Let $X \sim \text{Exponential}(\beta)$, for $\beta > 0$. Compute the median of X.

10. Two checkers at a service station complete checkouts independent of one another in times $T_1 \sim \text{Exponential}(\mu_1)$ and $T_2 \sim \text{Exponential}(\mu_2)$, respectively. That is, one checker serves $1/\mu_1$ customers per unit time on average, while the other serves $1/\mu_2$ customers per unit time on average.

    (a) Give the joint pdf, $f_{_{T_1,T_2}}(t_1, t_2)$, of T_1 and T_2.

    (b) Define the minimum service time, T_m, to be $T_m = \min\{T_1, T_2\}$. Determine the type of distribution that T_m has and give its pdf, $f_{_{T_m}}(t)$.

    (c) Suppose that, on average, one of the checkers serves 4 customers in an hour, and the other serves 6 customers per hour. On average, what is the minimum amount of time that a customer will spend being served at the service station?

11. Let T_1 and T_2 represent the lifetimes in hours of two linked components in an electronic device. The joint density function for T_1 and T_2 is uniform over the region defined by $0 \leqslant t_1 \leqslant t_2 \leqslant L$, where L is a positive constant. Determine the expected value of the sum of the squares of T_1 and T_2.

12. A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f(x, y) = \frac{x + y}{8}, \quad \text{for } 0 < x < 2 \text{ and } 0 < y < 2;$$

and 0 elsewhere.

What is the probability that the device fails during its first hour of operation?

13. A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E[X]$.

14. The number, Y, of spam messages sent to a server in a day has a Poisson distribution with parameter $\lambda = 21$. Each spam message independently has a probability $p = 1/3$ of not being detected by the spam filter. Let X denote the number of spam massages getting through the filter. Calculate the expected daily number of spam messages which get into the server.

15. Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation $0.0056h$. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation $0.0044h$.

Assuming the two measurements are independent random variables, what is the probability that their average value is within $0.005h$ of the height of the tower?

16. The lifetime of a printer costing \$200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

17. A company manufactures a brand of light bulb with a lifetime in months that is normally distributed with mean 3 and variance 1 . A consumer buys a number of these bulbs with the intention of replacing them successively as they burn out. The light bulbs have independent lifetimes. What is the smallest number of bulbs to be purchased so that the succession of light bulbs produces light for at least 40 months with probability at least 0.9772?

18. A computer manufacturing company conducts acceptance sampling for incoming computer chips. After receiving a huge shipment of computer chips, the company randomly selects 800 chips. If three or fewer nonconforming chips are found, the entire lot is accepted without inspecting the remaining chips in the lot. If four or more chips are nonconforming, every chip in the entire lot is carefully inspected at the supplier's expense. Assume that the true proportion of nonconforming computer chips being supplied is 0.001. Estimate the probability the lot will be accepted.

19. Last month your company sold 10,000 new watches. Past experience indicates that the probability that a new watch will need repair during its warranty period is 0.002.

    Estimate the probability that no more than 5 watches will need warranty work. Explain the reasoning leading to your estimate.

20. A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails.

    Let X and Y be the times at which the first and second circuits fail, respectively. Assume that X and Y have joint probability density function

$$f_{(X,Y)}(x,y) = \begin{cases} 6e^{-x}e^{-2y}, & \text{for } 0 < x < y < \infty; \\ 0, & \text{elsewhere.} \end{cases}$$

    What is the expected time at which the device fails?