Assignment #15

Due on Wednesday, October 29, 2014

Read Section 5.2 on *Marginal Distributions* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 5.3 on the *Independent Random Variables* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 3.9 on *Functions of Two or More Random Variables* in DeGroot and Schervish.

Read Section 4.10 on *Covariance and Correlation* in DeGroot and Schervish.

Background and Definitions

- **Definition of Covariance**. Given random variables X and Y, put $\mu_X = E(X)$ and $\mu_Y = E(Y)$. The *covariance* of X and Y, denoted $\text{Cov}(X,Y)$ is defined by
$$\text{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)], \tag{1}$$
provided that the expectation in (1) exists.

- **Definition of Correlation**. Let X and Y be random variables with finite variances σ_X^2 and σ_Y^2, respectively. The *correlation* of X and Y, denoted $\rho(X,Y)$, is defined by
$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}.$$

Do the following problems

1. Suppose X and Y are independent and let $g_1(X)$ and $g_2(Y)$ be functions for which $E(g_1(X)g_2(Y))$ exists. Show that
$$E(g_1(X)g_2(Y)) = E(g_1(X))\cdot E(g_2(Y))$$
Conclude therefore that if X and Y are independent and $E(|XY|)$ is finite, then
$$E(XY) = E(X)\cdot E(Y).$$

2. Suppose X and Y are independent random variables for which the moment generating functions exist on some common interval of values of t. Show that
$$\psi_{X+Y}(t) = \psi_X(t)\cdot\psi_Y(t)$$
for t is the given interval.

3. Let X and Y denote random variables for which $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ exist; that is, $\mathrm{Var}(X) < \infty$ and $\mathrm{Var}(Y) < \infty$. Show that $\mathrm{Cov}(X, Y)$ exists.

   Suggestion: Use the inequality

$$|ab| \leqslant \frac{1}{2}(a^2 + b^2),$$

   for all real numbers a and b.

4. Assume that X and Y have joint pdf

$$f_{(X,Y)}(x, y) = \begin{cases} 2xy + \dfrac{1}{2}, & \text{for } 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1; \\ 0, & \text{elsewhere.} \end{cases}$$

   Compute the covariance of X and Y.

5. Let X and Y denote random variables with finite variance.

   (a) Derive the identity

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y).$$

   (b) Show that if X and Y are independent, then

$$\mathrm{Cov}(X, Y) = 0 \quad \text{and} \quad \rho(X, Y) = 0.$$