Review Problems for Exam 2

(1) A random point (X, Y) is distributed uniformly on the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

(a) Give the joint pdf for X and Y.

(b) Compute the following probabilities: (i) $P(X^2+Y^2<1)$, (ii) $P(2X-Y>0)$, (iii) $P(|X+Y|<2)$.

(2) Let $F_{(X,Y)}$ be the joint cdf of two random variables X and Y. For real constants $a<b$, $c<d$, show that

$$\Pr(a<X\leq b, c<Y\leq d)=F_{(X,Y)}(b,d)-F_{(X,Y)}(b,c)-F_{(X,Y)}(a,d)+F_{(X,Y)}(a,c).$$

Use this result to show that

$$F(x,y)=\begin{cases}1 & \text{if } x+2y\geq 1,\\ 0 & \text{otherwise,}\end{cases}$$

cannot be the joint cdf of two random variables.

(3) The random pair (X, Y) has the joint distribution

X \ Y	2	3	4
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0

(a) Show that X and Y are not independent.

(b) Give a probability table for random variables U and V that have the same marginal distributions as X and Y, respectively, but are independent.

(4) Let X denote the number of trials needed to obtain the first head, and let Y be the number of trials needed to get two heads in repeated tosses of a fair coin. Are X and Y independent random variables?

(5) Prove that if the joint cdf of X and Y satisfies

$$F_{X,Y}(x,y)=F_X(x)F_Y(y),$$

then for any pair of intervals (a,b) and (c,d),

$$P(a<X\leq b, c<Y\leq d)=P(a<X\leq b)P(c<Y\leq d).$$

(6) Let $g(t)$ denote a non–negative, integrable function of a single variable with the property that

$$\int_0^\infty g(t)\ dt=1.$$

Define

$$f(x,y)=\begin{cases}\dfrac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}} & \text{for } 0<x<\infty,\ 0<y<\infty,\\ 0 & \text{otherwise.}\end{cases}$$

Show that $f(x,y)$ is a joint pdf for two random variables X and Y.

(7) Let $X \sim$ Exponential(1), and define Y to be the integer part of $X + 1$; that is, $Y = i + 1$ if and only if $i \leq X < i + 1$, for $i = 0, 1, 2, \ldots$ Find the pmf of Y, and deduce that $Y \sim$ Geometric(p) for some $0 < p < 1$. What is the value of p?

(8) Suppose that two persons make an appointment to meet between 5 PM and 6 PM at a certain location and they agree that neither person will wait more than 10 minutes for each person. If they arrive independently at random times between 5 PM and 6 PM, what is the probability that they will meet?

(9) Suppose that a book with n pages contains on average λ misprints per page. What is the probability that there will be at least m pages which contain more than k missprints?

(10) Suppose that the total number of items produced by a certain machine has a Poisson distribution with mean λ, all items are produced independently of one another, and the probability that any given item produced by the machine will be defective is p.
Let X denote the number of defective items produced by the machine.
 (a) Determine the marginal distribution of the random variable X.
 (b) Let Y denote the number of non–defective items produced by the machine. Show that X and Y are independent random variables.

(11) Suppose that the proportion of color blind people in a certain population is 0.005. Estimate the probability that there will be more than one color blind person in a random sample of 600 people from that population.

(12) An airline sells 200 tickets for a certain flight on an airplane that has 198 seats because, on average, 1% of purchasers of airline tickets do not appear for departure of their flight. Estimate the probability that everyone who appears for the departure of this flight will have a seat.

(13) Let X and Y denote random variables. Show that

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y).$$

Deduce that, if X and Y are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$